BCS-BEC crossover in Fermi superfluids and two-fluid hydrodynamics

The most interesting recent development in ultracold gases has been the study of superfluid **Fermi atomic gases**. What makes Fermi gases so interesting is that the strength of the inter-atomic interactions can be tuned easily by using a **Feshbach resonance**. Research on this topic has exploded since 2003.

In a two-component Fermi gas, by increasing the attractive interaction, one can **smoothly** go from a **BCS** phase (Fermi quasiparticles moving in a Cooper pair condensate) **BEC** phase (involving a Bose condensate of molecules). The Feshbach resonance allows us to study this crossover.

These **bound states** in **interacting** Fermi gases are **Bosonic** in nature and hence can **Bose condense**, just like Bose atoms can. As a result, in trapped Fermi gases, a sort of Bose condensate appears once again. It can describe a **molecular Bose condensate** or a **Cooper pair condensate,** immersed in the gas of unpaired Fermi atoms.

In the **extreme limit**, all N Fermi atoms form *N/2* bound states.This is the **BEC limit** of an interacting Fermi gas. It is effectively a Bose-condensed gas of *N/2* **molecules**, each with mass $M = 2m$.

This BCS-BEC crossover is beautiful physics. It has deepened our understanding of the **relation** between the BCS and BEC superfluids, showing that they are unified by the **role of a Bose condensate of bosonic pair states** in both cases. In the older literature, this BEC aspect of the BCS theory was **not** realized or emphasized.

It also allows us to produce a **strongly interacting Bose gas** of bosonic dimers composed of two Fermi atoms.

It gives us a superfluid Fermi gas with strong interactions which allow us to achieve **local hydrodynamic equilibrium**. When the scattering length is infinite (**unitarity**), we have strong collisions and hence the Landau two-fluid hydrodynamic equations will be valid. This gives a **new system where we can look for the characteristic first and second hydrodynamic oscillations.**

The BCS-BEC crossover is both interesting and complicated because the thermal excitations which determine the thermodynamic quantities involve both a **Fermi spectrum and a Bose spectrum.**

In the BCS weak coupling phase, we shall see that there are the expected **BCS fermi quasiparticles** plus the collective oscillations of the Cooper pair condensate. The latter are the **Anderson-Bogoliubov** Goldstone bosons.

As we go over to the BEC side, the BCS quasiparticles **disappear** as the Fermi atoms pair up to form real molecules. The end result is that are left with a gas of interacting bosonic molecules. The excitations of this molecular Bose condensate can be described in terms of **Bogoliubov excitations.**

All this will become clear(I hope!) in this lecture

Why work with a two-component Fermi gas?

- At ultra-low temperatures, atoms have very low momentum and hence only the **lowest partial wave** contributions from the interaction need be kept.

- Only the **s-wave scattering** contribution is large, but this does not arise between identical Fermions because of the Pauli principle. However, it can occur between atoms with **different** values of m_F (denoted by **spin**$\Uparrow$ and **spin** $\Downarrow$).

- This s-wave scattering allows rapid **thermalization** and hence cooling of the two-component Fermi gas.

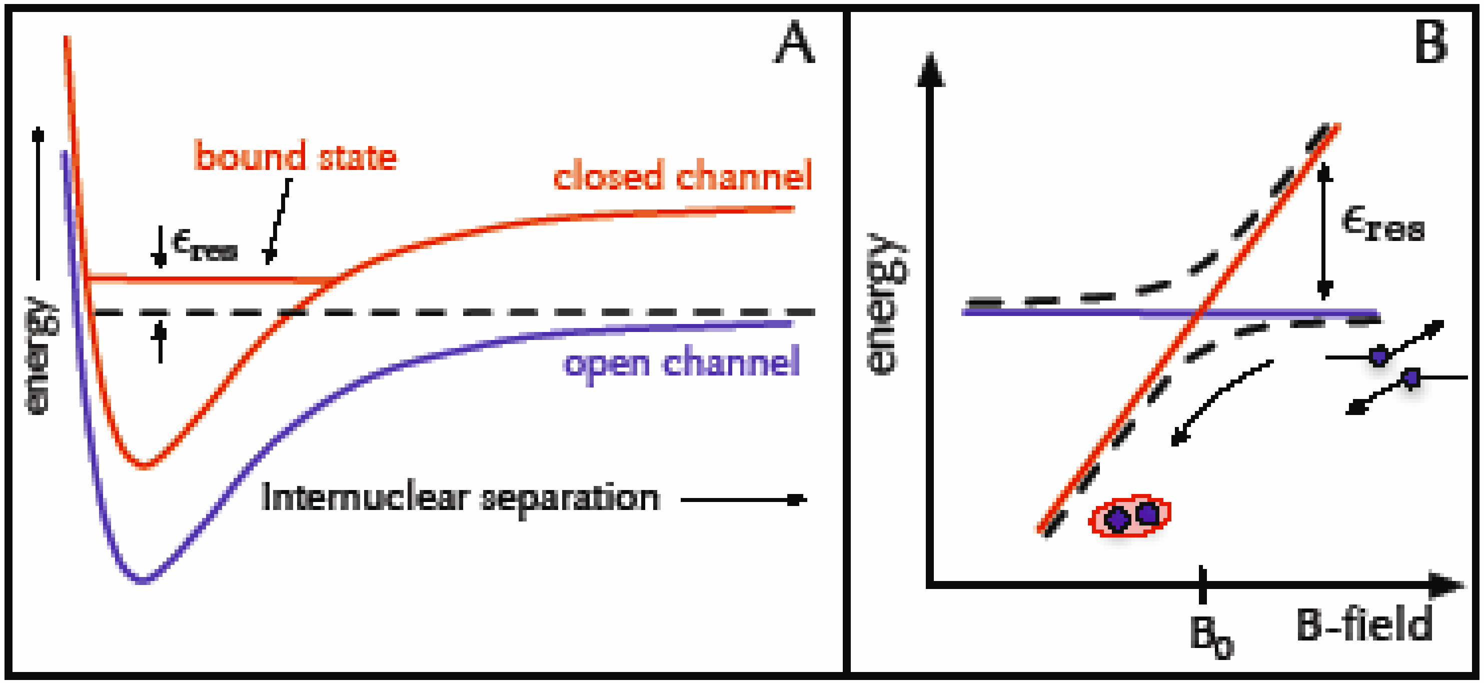
What are Feshbach Resonances?

- The key to creating the BCS-BEC crossover is the use of **Feshbach resonances** in the **atomic scattering cross-section**. These are a **two-body** phenomenon and exist in both **Bose** and **Fermi** gases. However, they are most useful in Fermi gases, for reasons we explain later.

- Such resonances arise when two colliding atoms have a total **kinetic energy** very close to the bound state energy level of a molecular potential (the so-called closed channel).

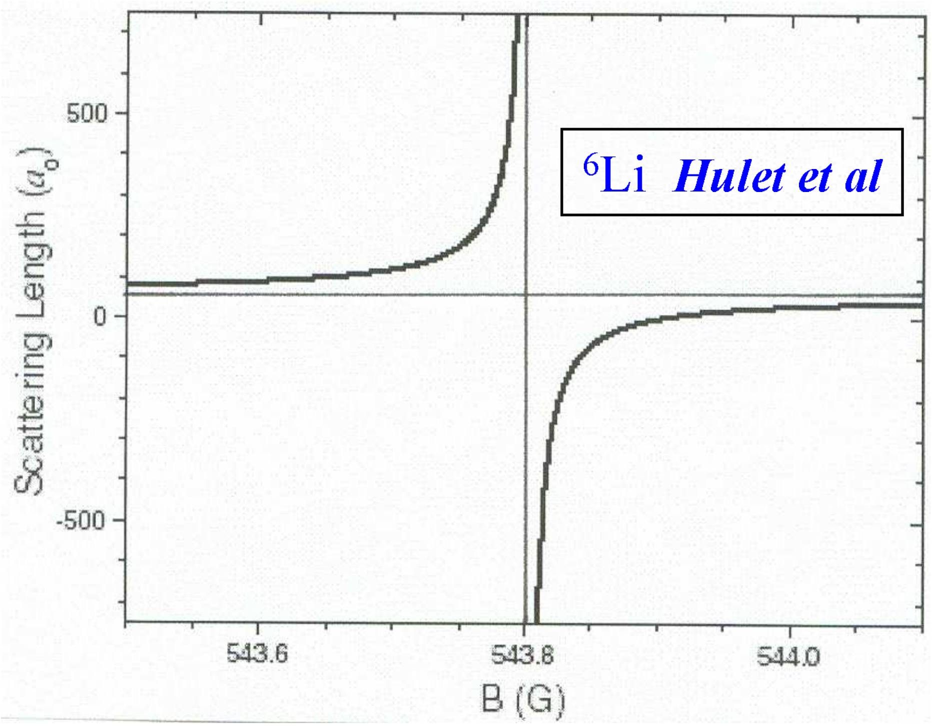
- The energy of the bound state molecular level can be shifted (**tuned**) by a small magnetic field B. The effective *s*-wave scattering length a_F has a **resonance** when the bound state has zero energy.

Two fermions in open channel strongly couple to a bound state, with energy ε_{res}



Two body physics of how dimer states can be created.
Process is reversible (adiabatic).

Feshbach resonance: two body physics

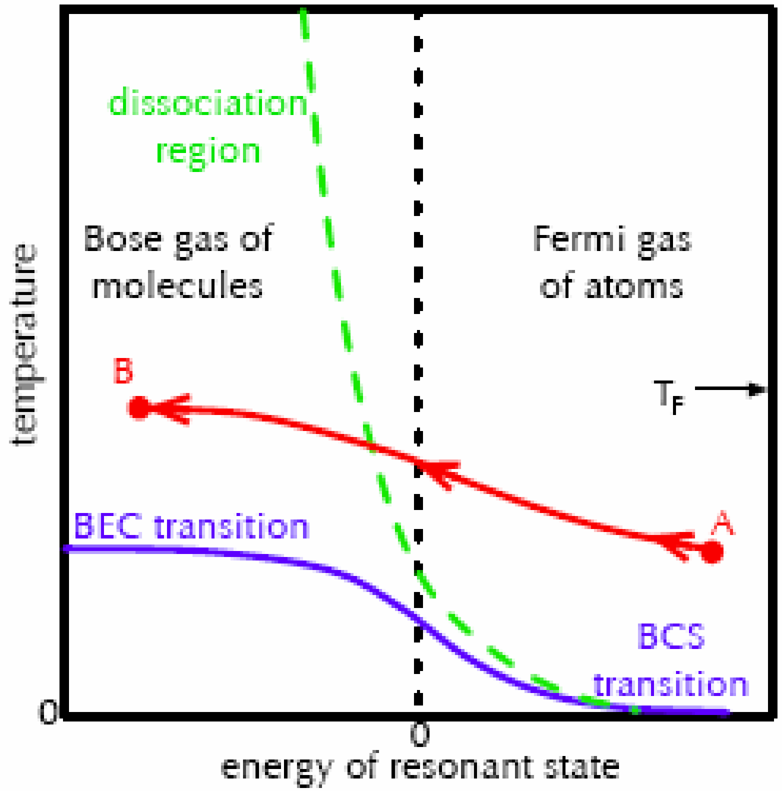


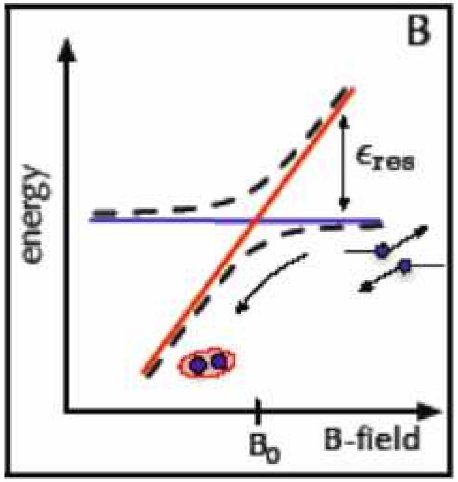
Molecules only form when $a_{2b} > 0$. This is equivalent to $\varepsilon < 0$ or $B < B_0$.

$$\varepsilon \propto B - B_0$$

$$a_S = a_{bg}\left(1 + \frac{w}{B_0 - B}\right)$$

These very weakly bound molecules on the BEC side interact with each other with a *s*-wave scattering length $a_M = 0.6\, a_F$ and are **very long-lived**. Since the dimer is composed of Fermi atoms in different Fermi states, an unpaired Fermi atom is repelled by **Pauli exclusion** and thus three body processes are suppressed. Almost by magic,we can produce a strongly interacting **molecular** Bose-condensed gas !





$$\varepsilon \propto B - B_0$$

The **blue curve** represents the **phase boundary** into the **superfluid state of bound pairs.**

BCS superfluid phase: a quick review

A two component Fermi gas (electrons in metals, ^{3}He atoms, alkali atoms) with an attractive interaction Is unstable to the **formation of a bound state of two Fermions** (of "opposite" spin). This Cooper pair is a many-body effect, and only arises in a **degenerate Fermi gas** . It does **NOT** depend on the **interatomic potential** having a real "bound state".

Once these Cooper pairs **(Bosons)** form at T_{BCS} , they produce a Cooper pair condensate. The remaining Fermi atoms swim around in this **condensate soup**, and develop a gap Δ in their **single particle** energy spectrum.

Using a pseudopotential for the s-wave interaction between spin up and spin down Fermi atoms,

$$v_{\uparrow\downarrow}(r-r') = \frac{4\pi\hbar^2 a_{\uparrow\downarrow}}{m}\delta(r-r') \equiv -U\delta(r-r')$$

$$V = \sum \int dr \int dr' \psi_{\downarrow}^{+}(r)\psi_{\uparrow}^{+}(r') v_{\uparrow\downarrow}(r-r')\psi_{\uparrow}(r')\psi_{\downarrow}(r)$$

$$= -U \int dr\, \psi_{\downarrow}^{+}(r)\psi_{\uparrow}^{+}(r)\psi_{\uparrow}(r)\psi_{\downarrow}(r)$$

$$= -U \sum_{p,q} c_{p\downarrow}^{+} c_{-p\uparrow}^{+} c_{-q\uparrow} c_{q\downarrow}$$

$$\boxed{\psi_{\alpha}(r) = \sum_{p} e^{ip.r} c_{p\alpha}}$$

$$H-\mu N=\sum_{p,\sigma}(\varepsilon_{p,\sigma}-\mu)c^{+}_{p,\sigma}c_{p,\sigma}-U\sum_{p,q}c^{+}_{p\uparrow}c^{+}_{-p\downarrow}c_{-q\downarrow}c_{q\uparrow}$$

$$\Rightarrow\Rightarrow\quad\approx\sum_{p,\sigma}(\varepsilon_{p,\sigma}-\mu)c^{+}_{p,\sigma}c_{p,\sigma}-\sum_{p}(\Delta c^{+}_{p\uparrow}c^{+}_{-p\downarrow}+h.c.)$$

$$\Delta=U\sum_{q}\langle c_{-q\downarrow}c_{q\uparrow}\rangle\equiv U\phi_C$$

Cooper pairs

This is the essence of the famous BCS-Gorkov theory of superconductivity in an interacting Fermi gases with an attractive interaction - *U*. The order parameter ϕ_C describes bound states of two Fermions, which are Bose-condensed into the same state. Remark:This MFA theory ignores Cooper pairs outside of the condensate.

Physics and math of BCS-Bogoliubov quasiparticles

One diagonalizes the BCS-Gorkov mean field Hamiltonian using the famous **Bogoliubov transformation**

$$c_{p\uparrow} = u_p \alpha_{p\uparrow} + v_{-p} \alpha^{+}_{-p\downarrow}$$

$$c_{p\downarrow} = u_p \alpha_{p\downarrow} - v_{-p} \alpha^{+}_{-p\uparrow}$$

The α, α^{+} quasiparticle operators to satisfy Fermi anti-commutation relations,such as

$$\left[\alpha_{p\uparrow}, \alpha^{+}_{q\uparrow}\right]_{+} = \delta_{p,q}$$

As a result, the Bogoliubov amplitudes ***u*** and ***v*** must satisfy the normalization condition

$$\left|u_{p\uparrow}\right|^2 + \left|v_{p\uparrow}\right|^2 = 1$$

We have reduced problem to a gas of **non-interacting** Fermi quasiparticles. Our favorite easy problem!

Calculation gives the following explicit expressions for the Bogoliubov u and v coefficients

$$u_p^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_p - \mu}{E_p}\right) \qquad v_p^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_p - \mu}{E_p}\right)$$

where the BCS quasiparticle **excitation energy** is

$$E_p = \left[(\varepsilon_p - \mu)^2 + \Delta^2\right]^{1/2}$$

This BCS mean field approximation is thus diagonalized by this Bogoliubov transformation:

$$H_{BCS} - \mu N = \sum_{p,\alpha} E_p \alpha^+_{p\alpha} \alpha_{p\alpha} + const.$$

The self-consistent equations for the BCS gap Δ

Clearly, we have two quantities we need to calculate using our quasiparticle solution, namely the chemical potential μ and the energy gap **Δ.** The **number equation** is

$$N \equiv \sum_{q,\alpha} \left\langle c^{+}_{q\alpha} c_{q\alpha} \right\rangle = \sum_{q} \left[\left|u_q\right|^2 \left\langle \alpha^{+}_{q\uparrow} \alpha_{q\uparrow} \right\rangle_{Bog} + \left|v_{-q}\right|^2 \left\langle \alpha_{-q\downarrow} \alpha^{+}_{-q\downarrow} \right\rangle_{Bog} \right]$$

using the fact that $\langle \alpha\alpha \rangle = 0$ and $\langle \alpha^{+}\alpha^{+} \rangle = 0$. Since the quasiparticles are non-interacting Fermions, $\langle \alpha^{+}\alpha \rangle = f(E)$ is the **Fermi distribution function** for quasiparticles. Thus

$$N = N_F = \sum_{q} \left[1 - \frac{\varepsilon_q - \mu}{E_q} + 2\frac{\varepsilon_q - \mu}{E_q} f(E_q) \right]$$

This is the BCS number equation, giving ***N*** as a function of **μ and Δ.**

The self-consistent BCS gap equation

We recall that the Cooper pair order parameter was defined as

$$\Delta \equiv U \sum_q \left\langle c_{-q\downarrow} c_{q\uparrow} \right\rangle \equiv U \phi_C$$

This can again be **easily calculated** by writing c and c^+ in terms of Bogoliubov quasiparticles, just as we did for the density,

$$\Delta = U \sum_q u_q v_q \left[\left\langle \alpha^+_{q\uparrow} \alpha_{q\uparrow} \right\rangle + \left\langle \alpha^+_{q\downarrow} \alpha_{q\downarrow} \right\rangle - 1 \right]$$

$$= U \sum_q u_q v_q \left(2 f(E_q) - 1 \right) = U \sum_q \frac{\Delta}{2E_q} \left(2 f(E_q) - 1 \right)$$

One last thing we have to do is **renormalize** the bare attractive interaction U to remove problems at high momentum. This turns out to be given by the two-body s-wave scattering length a_{2b}.

To summarize, the standard BCS theory reduces to two coupled equations for the number of fermions *N* and the gap function Δ (this is the BCS order parameter):

$$1 = -\frac{4\pi\hbar^2 a_{2b}}{m}\sum_q \left[\frac{1-2f(E_q)}{2E_q} - \frac{1}{2\varepsilon_q}\right]$$

$$N = N_F = \sum_q \left[1 - \frac{\varepsilon_q - \mu}{E_q} + 2\frac{\varepsilon_q - \mu}{E_q} f(E_q)\right]$$

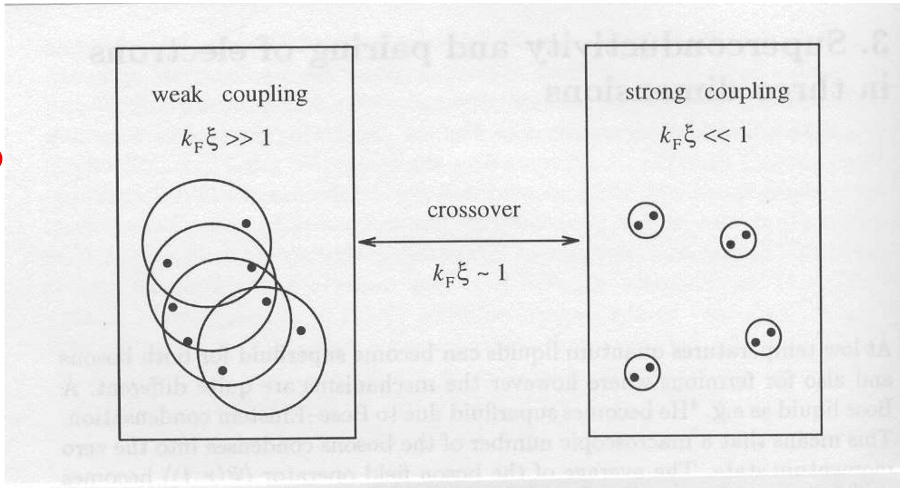
In standard **weak coupling** limit (when $k_F|a_{2b}| << 1$), there is no solution of the gap equation unless a_{2b} is **negative.** In this weak coupling BCS limit, one finds that $\mu \cong \varepsilon_F$. The BCS **transition temperature** is given by

$$T_{BCS} = T_F \exp\left(-\frac{\pi}{2k_F|a_{2b}|}\right) << T_F$$

The BCS-BEC Crossover -1980s

As the magnitude of the attractive interaction is increased, the Cooper pairs become more tightly bound and eventually we pass over to a region described as a dilute gas of small Cooper pair molecules. This is the famous BCS-BEC crossover, first studied in Eagles in 1969 and in the 1980s by Leggett (at T=0) and Nozieres (at T_c). At the same time, the spectral weight of the Fermi atoms decreases , as they combine to form Cooper pairs.

BCS



BEC

Somewhat surprisingly, BCS-BEC crossover at T = 0 can be studied simply by using these **full number and gap equations** And letting $k_F a_{2b}$ to be an adjustable parameter. Let the solutions tell us what happens! (Leggett, 1980)

It turns out that the dimensional parameter $(k_F a_{2b})^{-1}$ covers the range

$$-\infty \rightarrow \frac{1}{k_F a_{2b}} \rightarrow +\infty$$

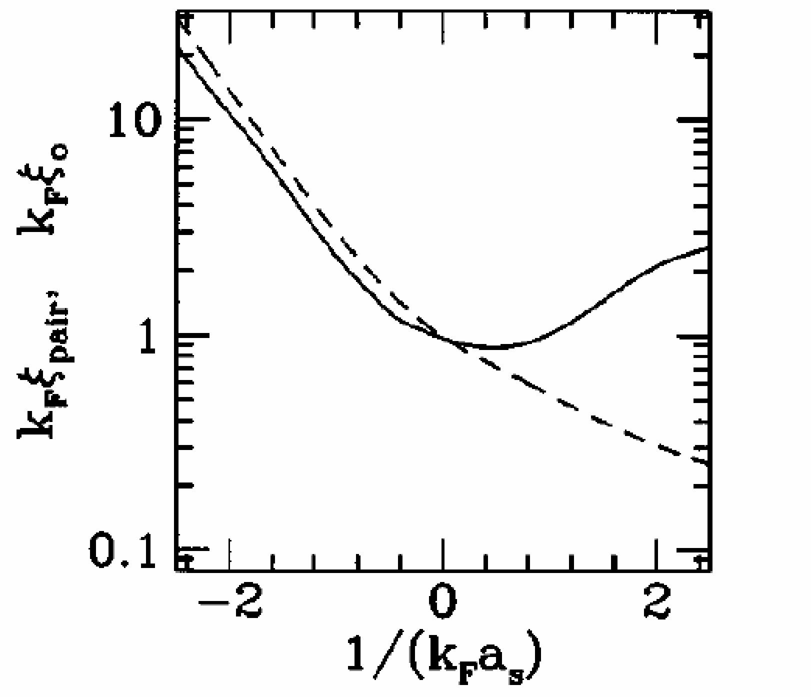
BCS **BEC**

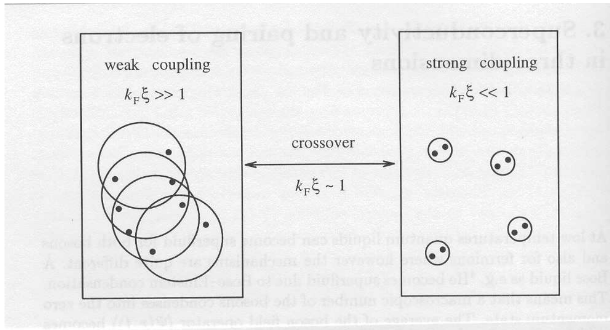
as the **bare attractive interaction is steadily increased**. These original calculations **did not address how** one could vary the value of the s-wave scattering length. **Feshbach resonances** allow you to do this easily in trapped atomic Fermi gases!!

As one crosses from the BCS side to the BEC side of the feshbach resonance, the **self-consistent value of the chemical potential** decreases from the Fermi energy down to zero, and then becomes large and negative. The BCS quasiparticle dispersion energy changes dramatically when the chemical potential is **negative and large**, since there is now large energy gap at $\boldsymbol{k = 0}$, not at the Fermi surface $k = k_F$ as on the BCS side

$$E_k = \left[(\varepsilon_k - \mu)^2 + \Delta^2\right]^{1/2}$$

When μ is large and negative, the Fermi surface is no longer important for the Fermi quasiparticles and the energy gap is **very large** and is given by |μ|, not by Δ. The Fermi quasiparticles get **frozen out**, letting the phonon modes dominate increasingly in the BEC side. These become the Bogoliubov phonons.

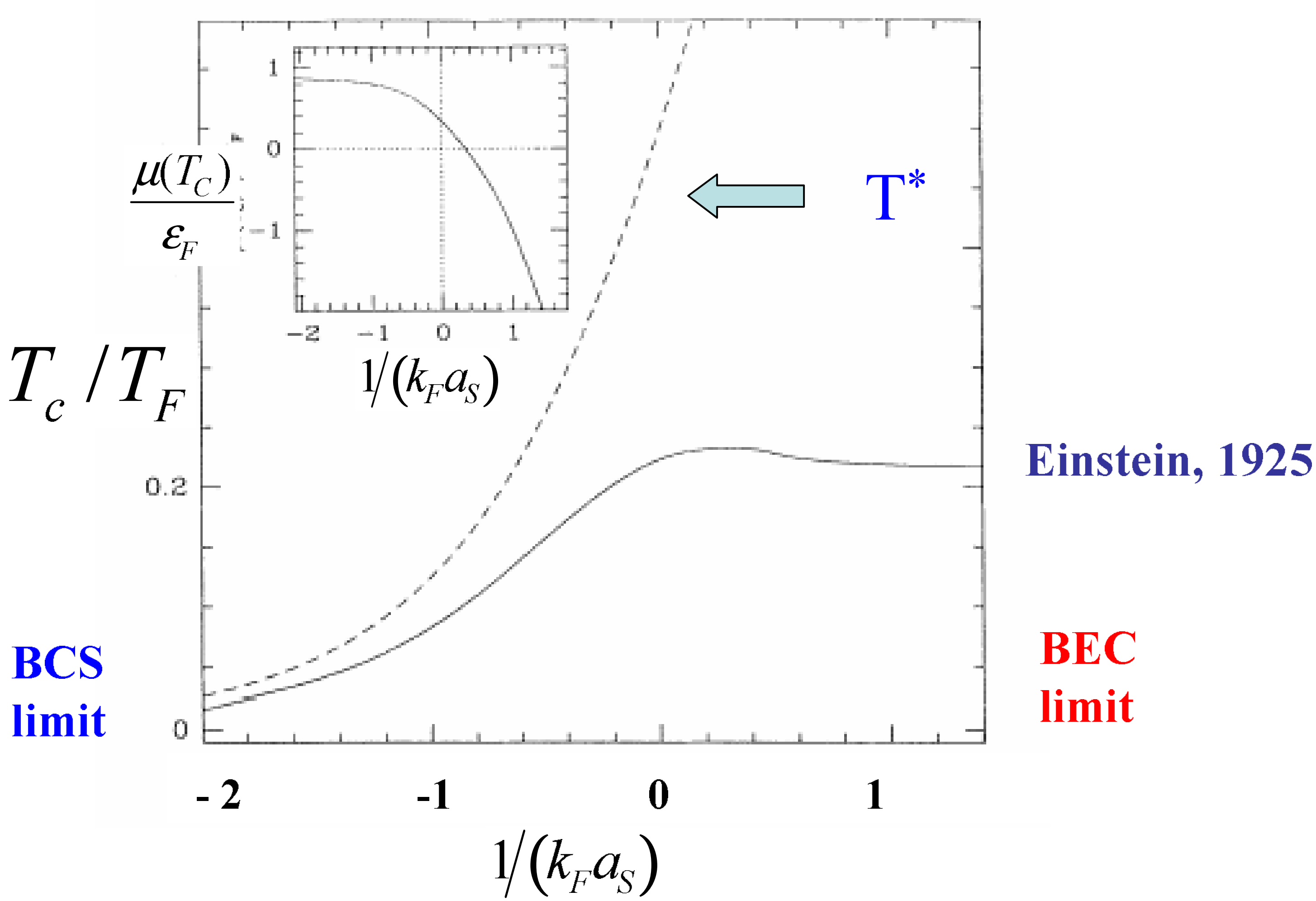




$$\frac{1}{k_F} \propto n^{-1/3} \approx d$$

Dashed line - - - - shows the smooth **decrease** in size of the bound state pair as we go from **BCS to BEC region.**

Engelbrecht, Randeria and Sa de Melo, PRB, 1997



T_C is the BCS-BEC superfluid phase transition temperature. T^* shows where the bound states **breakup** or ionize. Note that the **weak coupling** T_{BCS} result corresponds to the breakup of Cooper pairs, not depletion.

Sa de Melo , Randeria and Engelbrecht, PRL, 1993

A crucial bit of physics is left out of the BCS number and gap equations

When we think about it, our BCS equations **implicitly assume** that all the Cooper pairs are Bose condensed in the same center of mass momentum state, $q_{CM} = 0$ state. In the BCS number equation, we only calculated the contribution of the Fermi quasiparticles and **ignored** the dynamics of the pairs.

But it turns out that as the value of a_{2b} becomes > 0, the Cooper pairs become stable two-particle states and can **occupy finite momentum states**.Thus T is increased, more and more Cooper pairs leave the condensate. In this improved theory, T_C will correspond to where the bound **pair condensate is depleted**, just like the ideal Bose gas that Einstein considered in 1925! **Nozieres (1985).**

The method used by Nozieres and Schmitt-Rink (1985) replace the BCS number equation by calculating the number of Fermions using the **thermodynamic identity**

$$N \equiv -\frac{\partial \Omega}{\partial \mu}$$

where the thermodynamic potential $\Omega(\mu,T)$ of the interacting Fermi gas is given by:

$\Omega(\mu,T)$ = free energy of a **Fermi gas of atoms**

plus

Left out in BCS⇒

free energy of **fluctuations in the particle-particle channel**.These correspond to the formation of **bound states** of two atoms with finite center of mass momentum.

The calculation of all this involves a lot of many body theory, but the **esssential** physics is simple.

Equilibrium thermodynamics: thermal excitations through the crossover

- Need to know **equilibrium thermodynamic quantities**, including ρ_s and ρ_n, to calculate **hydrodynamic mode frequencies**.
- Calculation of thermodynamic quantities requires knowing what the **thermal (elementary) excitations** are in the crossover.
- *There are two types of thermal excitations*:

BEC limit: **phase modes** (phonons).

Close to unitarity, both modes exists. **They are strongly coupled.**

BCS limit: **BCS quasiparticles** from breakup of Cooper pairs.

We use the Nozières/Schmitt-Rink theory is used to include these excitations. **Pairing fluctuations beyond mean-field are needed to get the correct physics through the crossover.**

Once one has determined the the energy gap Δ and chemical potential μ of the BCS excitations in a **self-consistent fashion** taking into account the formation of pairs, the problem of determining the **thermodynamic potential** is reduced to a numerical integration.

The final thermodynamic potential reduces to the usual expression for a gas of non-interacting **Fermi and Bose excitations**,

$$\Omega = \frac{\Delta_0^2}{U} + \frac{1}{2}\sum \hbar\omega_q + \sum(\xi_k - E_k) - \frac{2}{\beta}\sum \ln\left(1 + e^{-\beta E_k}\right) + \frac{1}{\beta}\sum\left(1 - e^{-\beta\hbar\omega_q}\right)$$

in terms of the **BCS quasiparticle energies E_k** and the **bosonic mode energies ω_q** . One can obtain all thermodynamic quantities **needed** in the two-fluid equations from this kind of Ω .

Superfluid density in the BCS-BEC crossover

$$\rho_n^F = -\frac{2}{m}\int \frac{d^3\mathbf{k}}{(2\pi)^3}(\mathbf{k}\cdot\hat{\mathbf{Q}})^2\frac{\partial f(E_\mathbf{k})}{\partial E_\mathbf{k}}$$
$$= \frac{2}{3m}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathbf{k}^2\left(-\frac{\partial f(E_\mathbf{k})}{\partial E_\mathbf{k}}\right)$$

BCS limit

$$\rho_n^B = -\frac{2}{M}\int \frac{d^3\mathbf{q}}{(2\pi)^3}(\mathbf{q}\cdot\hat{\mathbf{Q}})^2\frac{\partial n_B(\omega_\mathbf{q})|}{\partial \omega_\mathbf{q}}$$

BEC limit

These two limits are well known. At unitarity, both **Fermi and Bose** excitations enter into the superfluid density

Second sound in Superfluid Fermi gases at unitarity

Using the previous theory, we have evaluated **all the thermodynamic functions which appear in the two-fluid equations**. We have done this for a uniform Fermi gas at unitarity, which is the region of greatest interest and where the Landau two-fluid equations are expected to be valid. This already requires extensive numerical work. We then use these uniform gas results to obtain the thermodynamic functions in a trap by using the standard local density approximation (LDA).

To find solutions of the Landau equations, we still have to solve these differential equations, where the **coefficients are dependent on position**. This is a difficult math problem! What we have done is to develop a **variational procedure** to solve these equations, starting from an ansatz of eight powers of ***r***, the coefficients being the variational parameters. This work was mainly done by Ed Taylor and Hui Hu. So far, we have only considered **breathing** two-fluid modes in an isotropic trap.

We believe our numerical results for the frequencies of first and second sound in a trapped gas are **quite accurate**, because we have also obtained analytic solutions for the second sound at **very low T** and also **near Tc**.

First and second sound in a strongly interacting Fermi gas

E. Taylor,[1,*] H. Hu,[2,3] X.-J. Liu,[2] L. P. Pitaevskii,[1,4] A. Griffin,[5] and S. Stringari[1]

[1]*CNR-INFM BEC Center and Dipartimento di Fisica, Università di Trento, I-38050 Povo, Trento, Italy*

[2]*ACQAO and Centre for Atom Optics and Ultrafast Spectroscopy, Swinburne University of Technology, Melbourne, Victoria 3122, Australia*

[3]*Department of Physics, Renmin University of China, Beijing 100872, China*

[4]*Kapitza Institute for Physical Problems, Kosygina 2, 119334 Moscow, Russia*

[5]*Department of Physics, University of Toronto, Toronto, Ontario, Canada, M5S 1A7*

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Many classic signatures of superfluidity have been observed in ultracold gases including quantized vortices, absence of viscosity, and the Josephson effect. Arguably the most dramatic manifestation of superfluidity, second sound has not yet been observed in cold gases, however. In

dynamics in a strongly interacting Fermi gases will be crucial to measuring quantities of interest such as transport coefficients [6, 11], the superfluid density [12], and the finite temperature equation of state [13, 14].

The dissipationless Landau two-fluid equations in a trap $V_{\rm ext}$ are given by [2, 8, 15]:

Submitted to **Nature Physics** just two days ago!
This paper has taken us about four year of work.
It grew out of Ed Taylor's Ph.D thesis at Toronto

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \left(\frac{\partial P}{\partial \rho}\right)_{\bar{s}} \nabla^2 \delta\rho + \left(\frac{\partial P}{\partial \bar{s}}\right)_{\rho} \nabla^2 \delta\bar{s} + n_0 \nabla^2 \delta U$$

$$\frac{\partial^2 \delta\bar{s}}{\partial t^2} = \bar{s}_0{}^2 \frac{\rho_{s0}}{\rho_{n0}} \left[\left(\frac{\partial T}{\partial \rho}\right)_{\bar{s}} \nabla^2 \delta\rho + \left(\frac{\partial T}{\partial \bar{s}}\right)_{\rho} \nabla^2 \delta\bar{s}\right]$$

Linear response gives the density fluctuation to first order in the perturbation. For a uniform gas, this is easily found:

$$\delta\rho(\mathbf{q},\omega) = n_0 q^2 \frac{\omega^2 - \bar{s}^2 \frac{\rho_{s0}}{\rho_{n0}} \left(\frac{\partial T}{\partial \bar{s}}\right)_{\rho} q^2}{(\omega^2 - u_1^2 q^2)(\omega^2 - u_2^2 q^2)} \delta U(\mathbf{q},\omega)$$

where the sound velocities are the solutions(see **Lecture 1**)

$$u^4 - u^2 \left[\left(\frac{\partial P}{\partial \rho}\right)_{\bar{s}} + \frac{\rho_{s0}}{\rho_{n0}} \frac{T\bar{s}_0^2}{\bar{c}_v}\right] + \frac{\rho_{s0}}{\rho_{n0}} \left(\frac{T\bar{s}_0^2}{\bar{c}_v}\right) \left(\frac{\partial P}{\partial \rho}\right)_T = 0.$$

Note that both **first and second sound** appear as **poles** of the density response function, but with quite **different** weights.

One can insert these solutions into the Landau equations and check:

First sound is a simple in-phase motion of two components

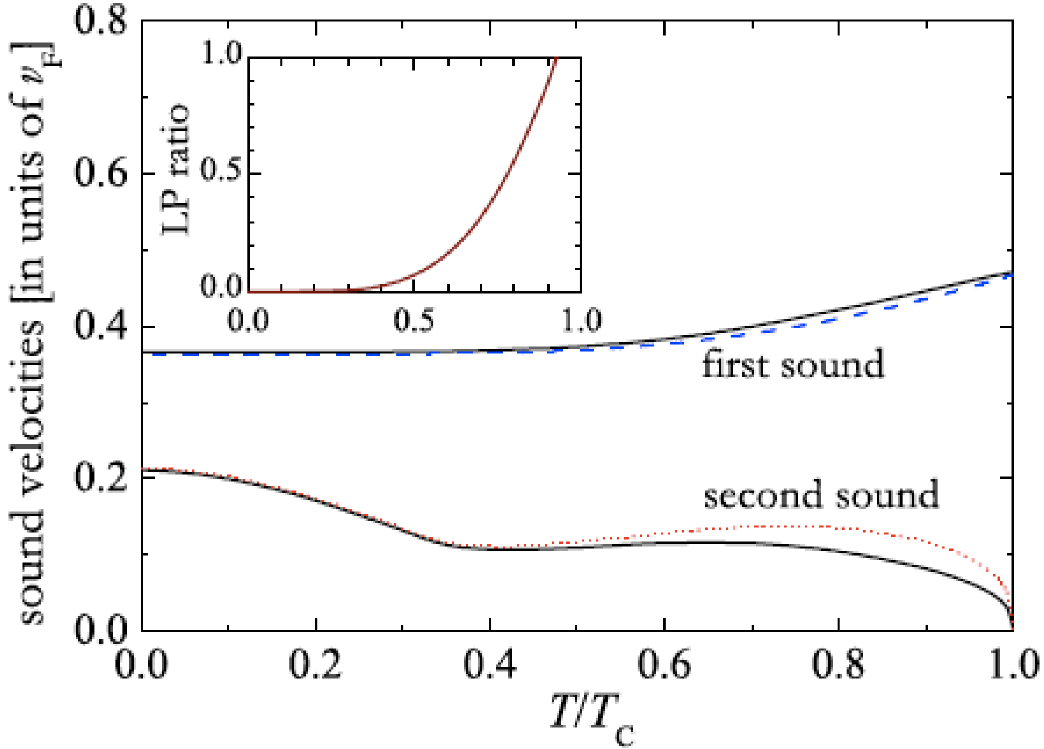
$$\delta\vec{v}_s = \delta\vec{v}_n \quad and\ \delta T = 0$$

Second sound is an out-of -phase motion of two components

$$\rho_{s0}\vec{v}_s = -\rho_{n0}\vec{v}_n \quad i.e.\ \delta\vec{j} = 0\ and\ thus\ \delta\rho = 0$$

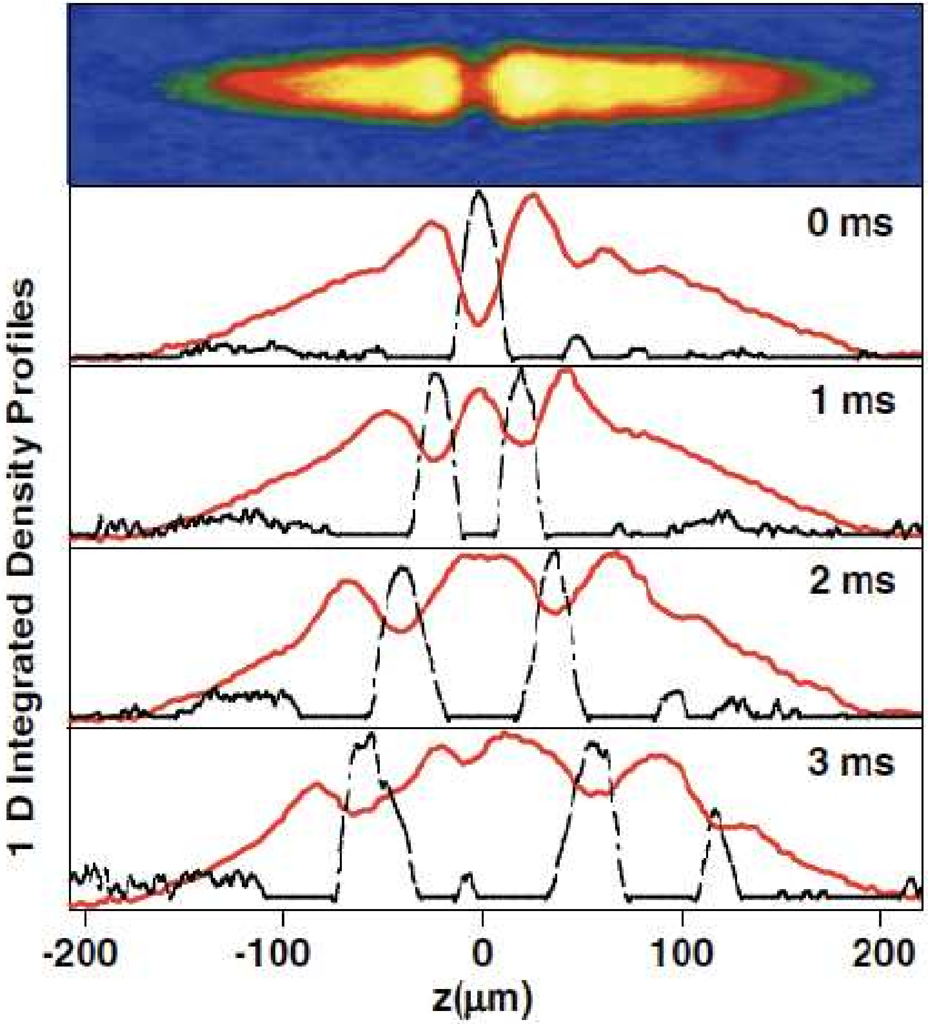
In other words, the modes are very similar in nature to those in superfluid ^{4}He.

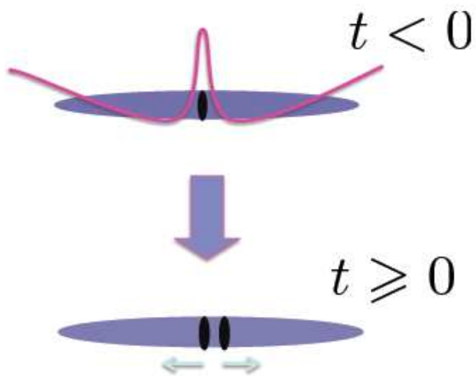
First and second sound velocities in a uniform Fermi gas at unitarity



At **low T**, the bosonic phonons dominate the thermodynamics
At **higher temperatures**, the Fermi quasiparticles are most important thermal excitations (they play the role as rotons in He.

Pulse propagation in long cigar-shaped traps



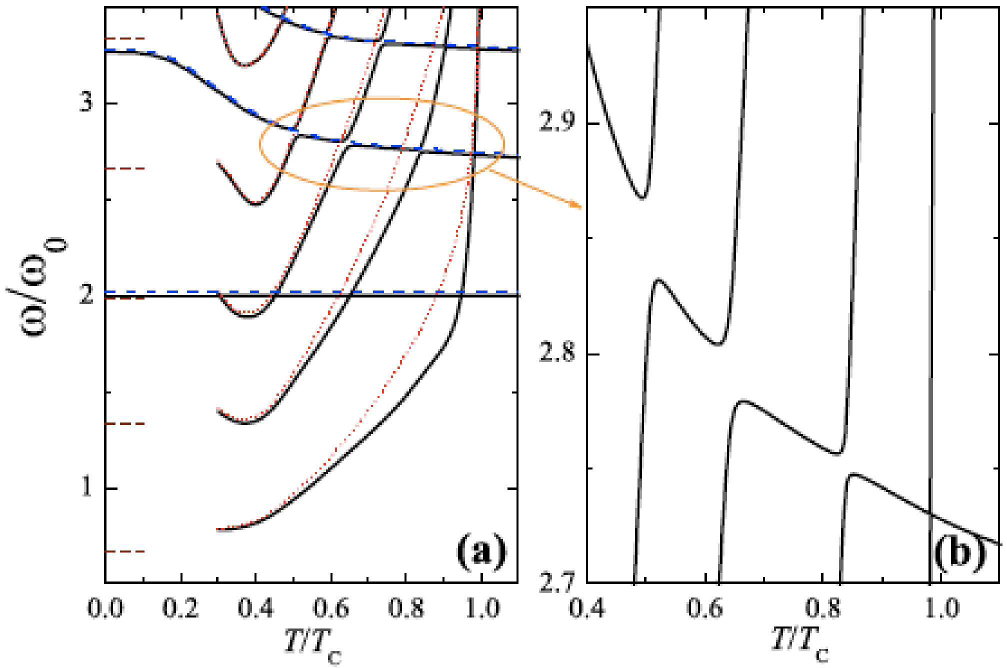


$$\delta U(z,t) = \delta U(z)\theta(-t)$$

Experiments by the group of John Thomas at Duke University, USA

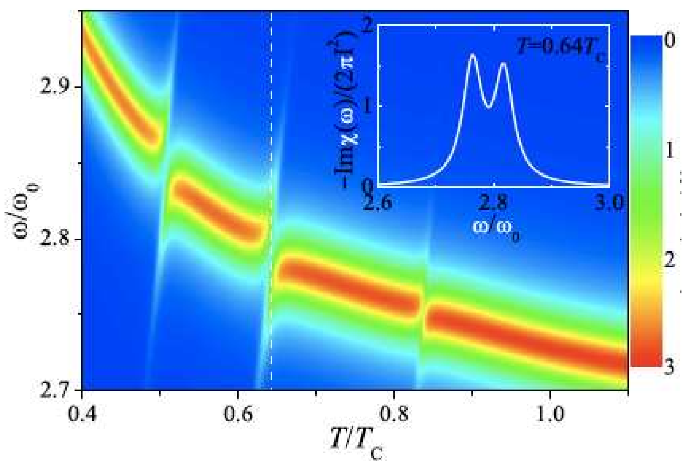
These results are done at unitarity in Fermi gas but at very low T

First and second sound frequencies at unitarity in a trapped gas



The **left side** shows the spectrum of first and second sound breathing modes, as a function of T

The **horizontal** lines are first sound branches and the **vertical** lines are second sound.



To date, the **only** breathing mode studied by experiment is the **lowest frequency** first sound mode. This happens to be independent of temperature.

Thank you for inviting me to this Winter School!

Thank you for listening to me talk about my research!

Adios!