

### Two-fluid hydrodynamics in a Bose-condensed gas

We now turn to **extending** the  $T = 0$  theory we discussed this morning to finite temperatures by including the effect of the **thermal cloud atoms**. Our starting point is the **exact** equation for the macroscopic condensate wavefunction:

$$i\hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) \right] \Phi(\mathbf{r}, t) + g \langle \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \rangle$$

We next split the quantum field operators into a condensate and noncondensate part (Beliaev, 1958)

$$\hat{\psi}(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \tilde{\psi}(\mathbf{r}, t)$$

The noncondensate part describes the **thermal cloud** atoms.

Using this decomposition in the correlation function with three field operators, we obtain

$$\hat{\psi}^\dagger \hat{\psi} \hat{\psi} = |\Phi|^2 \Phi + 2|\Phi|^2 \tilde{\psi} + \Phi^2 \tilde{\psi}^\dagger + \Phi^* \tilde{\psi} \tilde{\psi} + 2\Phi \tilde{\psi}^\dagger \tilde{\psi} + \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi}.$$

Taking the average reduces this to

$$\langle \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \rangle = n_c \Phi + \tilde{m} \Phi^* + 2\tilde{n} \Phi + \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle$$

and hence

$$i\hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) + gn_c(\mathbf{r}, t) + 2g\tilde{n}(\mathbf{r}, t) \right] \Phi(\mathbf{r}, t) + g\tilde{m}(\mathbf{r}, t) \Phi^* + g \langle \tilde{\psi}^\dagger(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \rangle.$$

We see this equation goes **past the original GP** by having **new terms** related to the noncondensate field operator  $\tilde{\psi}$

The first line includes the new **Hartree-Fock mean field** of the thermal cloud acting on the condensate, with density

$$\tilde{n}(\mathbf{r}, t) \equiv \langle \tilde{\psi}^\dagger(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \rangle$$

We are now going to make some approximations, leading to a theory developed by Zaremba, Nikuni and Griffin in 1999- called **ZNG for short**. These approximations are supposed to be good if we can use the **semiclassical** model of the thermal gas. This means that thermal gas atoms of momentum  $\mathbf{p}$  move in the trap and the HF self-consistent mean fields, with energy

$$\begin{aligned}\tilde{\varepsilon}_p(\mathbf{r}, t) &= \frac{p^2}{2m} + V_{\text{trap}}(\mathbf{r}) + 2g [n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)] \\ &\equiv \frac{p^2}{2m} + U(\mathbf{r}, t).\end{aligned}$$

Note the potential energy  $U(\mathbf{r}, t)$  depends only on  $\mathbf{r}$  and  $t$ . This semiclassical approximation ignores the **discrete quantum energy levels** of the harmonic trap. It is only valid at high enough temperatures (but below  $T_c$ ) such that

$$k_B T \gg gn_c, \quad k_B T \gg \hbar\omega_0.$$

But this is the region mainly studied in Bose gas experiments.

## Elementary excitations vs collective modes

It is very important in my discussion to keep clearly in mind the difference between:

- **Elementary excitations** (quasiparticles) which determine the thermodynamic properties of the system.
- **Collective modes** which involve oscillations in the density of the elementary excitations. These include the two-fluid hydrodynamic modes( first and second sound)

A good example to keep in mind is the **air in this room**. The elementary excitations are well approximated as an ideal gas of **free atoms**. In addition, we know that this Air sustains low frequency hydrodynamic **sound waves**, as predicted by the hydrodynamic equations. The sound velocity of these collective density oscillations is given by the usual **compressibility** of an ideal gas of atoms.


To complete the ZNG theory approximations, we ignore the anomalous correlation function  $\tilde{m}(\mathbf{r}, t) \equiv \langle \psi(\mathbf{r}, t) \psi(\mathbf{r}, t) \rangle$ , and we evaluate the three field correlation function involving the  $\tilde{\psi}$  field operators of the non-condensate thermal gas atoms using the previously defined semiclassical model to obtain

$$\begin{aligned} \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle = & -\frac{ig\Phi}{(2\pi)^5 \hbar^6} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) \\ & \times [f_1(1 + f_2)(1 + f_3) - (1 + f_1)f_2f_3]. \end{aligned}$$

where  $f_i = f(\mathbf{p}_i, \mathbf{r}, t)$  is the **single particle distribution function for atoms with momentum  $\mathbf{p}_i$** . Note the energy and momentum conservation delta functions. Also that it is first order in the interaction  $g$  and is imaginary.

**I will not derive this here, but verify its correctness shortly.**

If we insert these results in our exact equation for  $\Phi(r, t)$ , we obtain

$$i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) + gn_c(\mathbf{r}, t) + 2g\tilde{n}(\mathbf{r}, t) - iR(\mathbf{r}, t) \right] \Phi$$


where the new imaginary term is given by

$$R(\mathbf{r}, t) \equiv \frac{\hbar \Gamma_{12}[f, \Phi]}{2n_c(\mathbf{r}, t)}$$

in terms of the new **source function** associated with coupling to the thermal cloud atoms:

$$\begin{aligned} \Gamma_{12}(\mathbf{r}, t) = & 2 g^2 \frac{n_c(\mathbf{r}, t)}{(2\pi)^5} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(\mathbf{p}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times \delta(\varepsilon_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) \\ & \times [f_1(1 + f_2)(1 + f_3) - (1 + f_1)f_2f_3]. \end{aligned}$$

## *Kinetic equation for the dynamics of the thermal atoms*

$f(\mathbf{p}, \mathbf{r}, t)$  describes the single-particle distribution for atoms of momentum  $\mathbf{p}$ , at position  $\mathbf{r}$  and time  $t$ . A kinetic equation describes how it change in time when perturbed from equilibrium:

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \vec{\nabla}_r f \cdot \frac{\partial \mathbf{r}}{\partial t} + \vec{\nabla}_p f \cdot \frac{\partial \mathbf{p}}{\partial t} = \frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \vec{\nabla}_r f + \vec{F} \cdot \vec{\nabla}_p f \\ &= C_{12}[f, \Phi] + C_{22}[f, \Phi]\end{aligned}$$

Atoms move in a **self-consistent time-dependent Hartree-Fock potential**

$$\tilde{\mathcal{E}}_p(\mathbf{r}, t) = \frac{p^2}{2m} + U(\mathbf{r}, t)$$

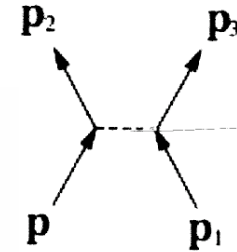
$$U(\mathbf{r}, t) \equiv V_{\text{trap}}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)]$$

$$\vec{F} \equiv \frac{\partial \mathbf{p}}{\partial t} = -\vec{\nabla}_r U(\mathbf{r}, t)$$

## Two kinds of collision terms in the kinetic equation

Collisions between thermal atoms are

$$C_{22}[f] = \frac{2g^2}{(2\pi)^5 \hbar^7} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4 \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \\ \times \delta(\tilde{\epsilon}_p + \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3} - \tilde{\epsilon}_{p_4}) [(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)]$$



Note we are taking into account of **Bose statistics**. In scattering, we have the statistical factors  $(1+f_i)$  for the creation or  $f_i$  for the destruction of an atom in state  $i$ . For Bose atoms,  $f_i$  can be **large**.

This type of collision integral reduces to the usual one for a **classical gases**, where all the distribution functions satisfy  $f \ll 1$ . In this case, the terms in the last square bracket reduce to

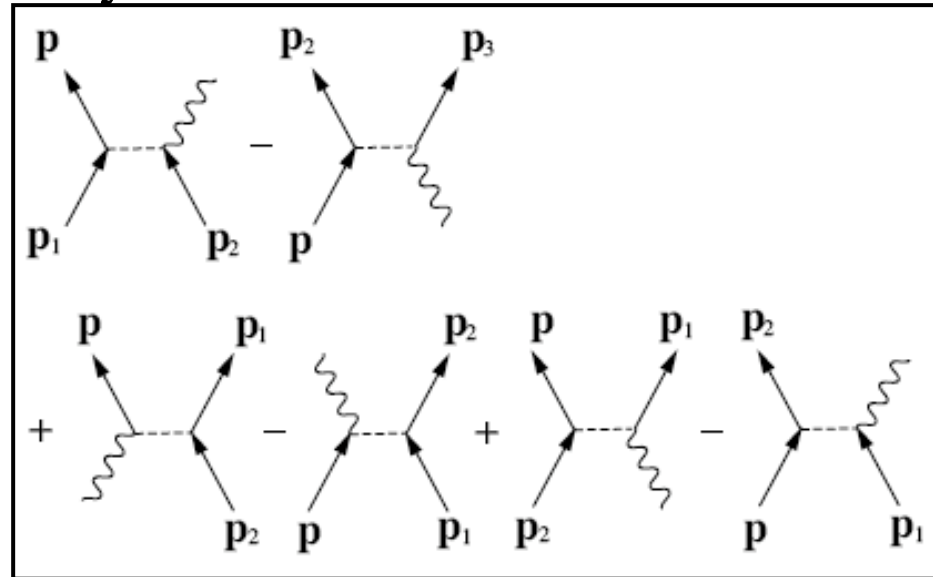
$$[f_3 f_4 - f f_2]$$

We note that the collision integrals are **second order** in the interaction strength  $g$  (binary collisions since the gas is dilute).



The **new** kind of collision term only arises when there is a Bose condensate involves **scattering between atoms in the condensate and thermal cloud**. These collisions play a crucial in the whole theory.

**$C_{12}$  collisions**



wiggly lines are condensate atoms

$$\begin{aligned}
 C_{12}[f] = & \frac{2g^2 n_c}{(2\pi)^2 \hbar^4} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\
 & \times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] \\
 & \times [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)],
 \end{aligned}$$

We note that the expression for  $C_{12}$  collision integral in our kinetic equation is **essentially the same** as the imaginary term  $R$  we found in our generalized GP (GGP) equation, namely

$$R(\mathbf{r}, t) \equiv \frac{\hbar}{2n_c(\mathbf{r}, t)} \Gamma_{12}[f, \Phi] = \frac{\hbar}{2n_c(\mathbf{r}, t)} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} C_{12}[f(\mathbf{p}, \mathbf{r}, t), \Phi(\mathbf{r}, t)]$$

This connection is not unexpected. We now show what the **precise meaning of this connection, in more physical terms.**

The condensate generalized GP (GGP) equation can be rewritten in terms of **density** and **velocity** variables. Doing exactly the same kind of math that we did for the GP equation, we find

$$\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_c) = -\Gamma_{12}[f]$$

$$m \left( \frac{\partial}{\partial t} + \mathbf{v}_c \cdot \nabla \right) \mathbf{v}_c = -\nabla \mu_c,$$

Using the relation

$$R(\mathbf{r}, t) \equiv \frac{\hbar \Gamma_{12}[f, \Phi]}{2n_c(\mathbf{r}, t)}$$

$$\Gamma_{12}(f, \Phi) \equiv \int \frac{dp}{(2\pi\hbar)^3} C_{12}[f, \Phi]$$

$$\mu_c(\vec{r}, t) = V_{trap}(\vec{r}) + gn_c(\vec{r}, t) + 2g\tilde{n}(\vec{r}, t)$$

As a result of the  $C_{12}$  collisions, the conservation equation for the condensate now has a **source term**  $\Gamma_{12}$  by which the atoms from the thermal cloud can **modify** the local number density of condensate atoms. **All this makes sense!**

To show our theory is consistent, let us return to our kinetic equation for the thermal atoms.

$$\frac{\partial f}{\partial t} + \vec{\nabla} \cdot \frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \vec{\nabla}_r f + \vec{F} \cdot \vec{\nabla}_p f = C_{12} + C_{22}$$

We now **integrate over the momentum** . We use the fact that that  $C_{22}$  collisions do **not change** the total number of atoms in the thermal cloud and hence

$$\int d\vec{p} C_{22}[f, \Phi] = 0$$

However, the  $C_{12}$  collisions **may change** the number of atoms in the thermal cloud and hence in general

$$\int d\vec{p} C_{12}[f, \Phi] \neq 0 \quad \therefore \Gamma_{12} \neq 0$$

Integrating over the momentum, the above kinetic equation above gives

$$\frac{\partial}{\partial t} \int \frac{d\vec{p}}{(2\pi\hbar)^3} f + \int \frac{d\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}}{m} \cdot \nabla_r f - \nabla_r U \cdot \int \frac{d\vec{p}}{(2\pi\hbar)^3} \nabla_p f = 0 + \int \frac{d\vec{p}}{(2\pi\hbar)^3} C_{12}$$

or

$$\frac{\partial \tilde{n}}{\partial t} + \nabla_r \cdot (\tilde{n} \vec{v}_n) - \nabla_r U \cdot \int \frac{d\vec{p}}{(2\pi\hbar)^3} \nabla_p f(\vec{r}, \vec{p}, t) = \Gamma_{12}(\vec{r}, t)$$



This integral vanishes

The thermal cloud local **density** and **velocity** variables are defined as

$$\tilde{n}(\mathbf{r}, t) \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f(\mathbf{p}, \mathbf{r}, t)$$

$$\mathbf{v}_n(\mathbf{r}, t) \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} \frac{f(\mathbf{p}, \mathbf{r}, t)}{\tilde{n}(\mathbf{r}, t)}$$

Thus the conservation equation for the local thermal cloud density has the **same source term** as the condensate conservation equation, but with opposite (or + ) sign.

$$\frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\tilde{n} \mathbf{v}_n) = \Gamma_{12}[f, \Phi]$$

Adding these two **separate conservation** equations, we see that  $\Gamma_{12}$  cancels out, and we obtain the correct conservation equation for the total **density** and **current**

$$\boxed{\frac{\partial n}{\partial t} + \nabla_r \cdot \vec{j} = 0}$$

where

$$n = n_c + \tilde{n}$$

$$\vec{j} = n_c \vec{v}_c + \tilde{n} \vec{v}_n$$

# Summary: ZNG coupled equations

Generalized GP equation for the condensate dynamics

$$i\hbar \frac{\partial \Phi}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V + g[n_c + 2\bar{n}] - iR \right) \Phi$$

is coupled to the kinetic equation for the thermal cloud

mean field coupling

Collisional coupling  
(atom transfer)

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - \nabla U \cdot \nabla_{\mathbf{p}} f = C_{22}[f] + C_{12}[f]$$

$$C_{12} \text{ collisions} \Rightarrow R = \frac{\hbar}{2n_c} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} C_{12}$$

These two coupled equations can be solved for the dynamics of the condensate and the thermal cloud at finite temperatures. Surprisingly the coupled dynamics contained in these simplified ZNG equations is still quite **subtle and complicated**.

Our aim is to use them to “unpack” the dynamics of trapped Bose gases at finite temperatures, especially on the **role of the  $C_{12}$  collisions**. There are two regions of interest when considering collective oscillations with frequency  $\omega$  :

**Collisionless region** - where the dominant effects came from the self-consistent **mean fields**. The effect of collisions are important but are usually perturbative corrections. Defined by  $\omega\tau \gg 1$

**Two-fluid collisional hydrodynamics-** where one achieves **local equilibrium due to strong collisions**  $\omega\tau \ll 1$



## Numerical simulations of ZNG equations in the collisionless region

Jackson and Zaremba have used Monte Carlo numerical simulations to solve the ZNG kinetic equation coupled to the GGP equation. See Chs.11 and 12 of the GNZ book for details.

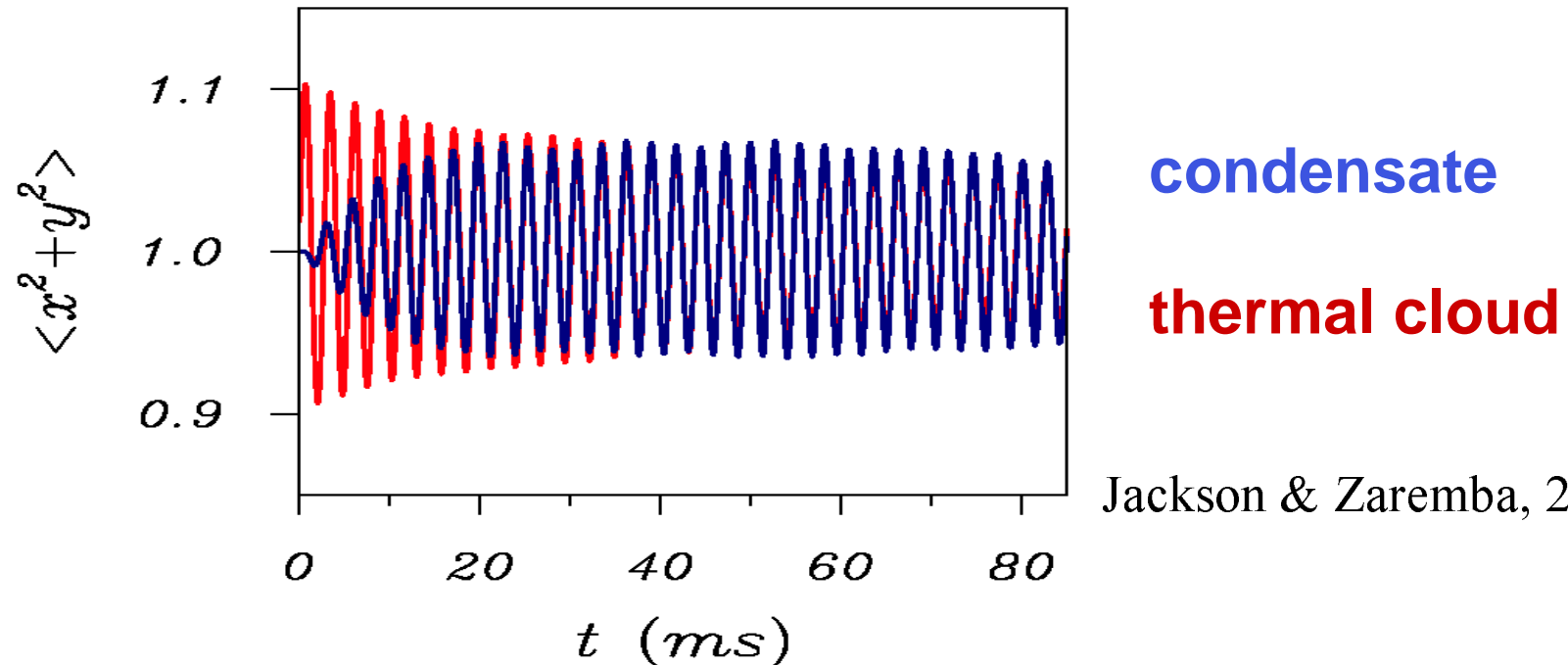
In this approach, one can **simulate the precise experimental conditions** used to excite the collective oscillations. One is not limited to **linear response**.

Agreement with data for the **temperature dependence of the frequency and damping** of the condensate and thermal cloud oscillations is **excellent**.

However, we note that while collisions are important in this region, they are not strong enough to produce local equilibrium

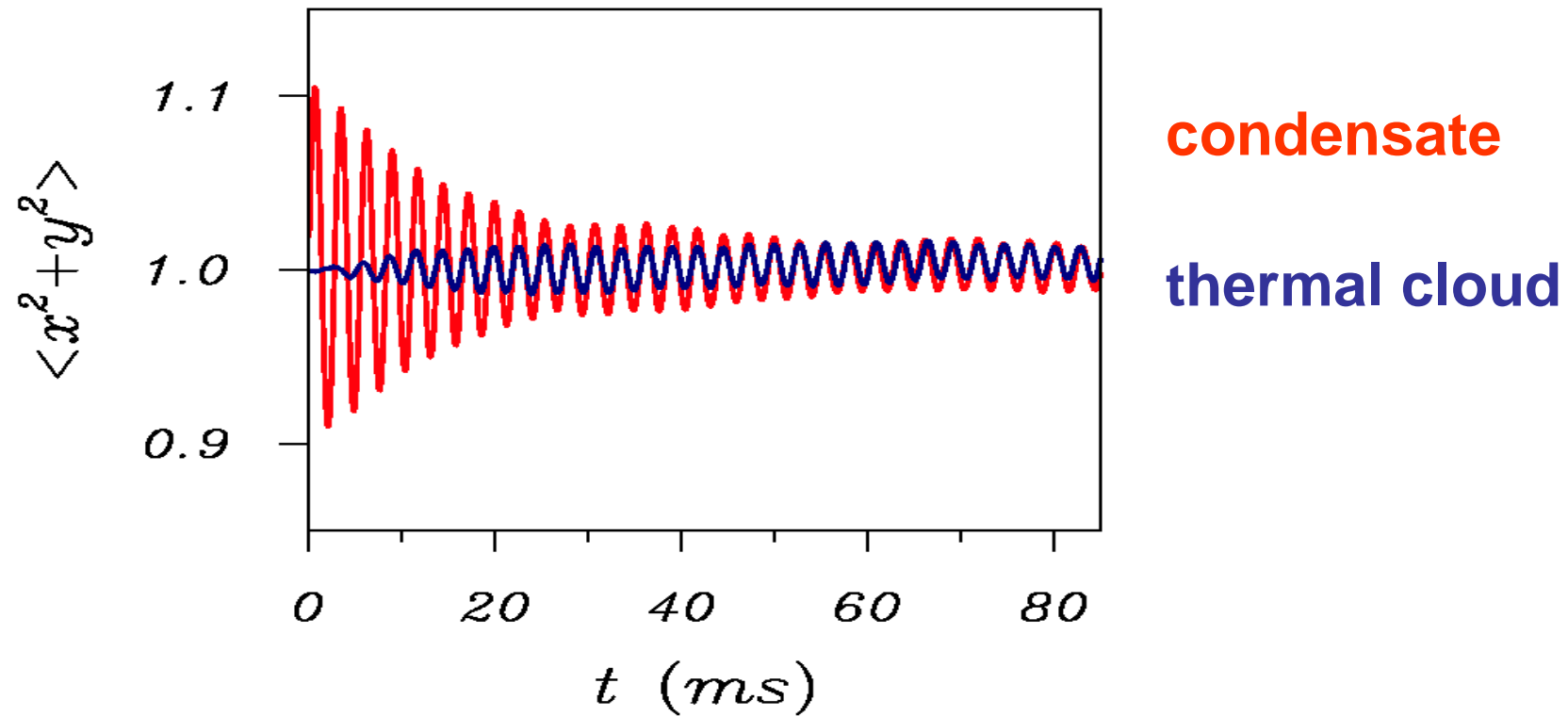
## Two examples of direct simulation of coupled dynamics

*Initial conditions: Excite thermal cloud only*



From such **direct simulations** of the cloud dynamics, one can obtain the **frequency** and **damping** of the collective modes as a function of the **temperature**. This simulation was done to explain experiments by Dalibard group, ENS, 2002.

*Initial conditions: Excite condensate only*



Jackson & Zaremba, 2002

## Static thermal cloud approximation

The **simplest theory** which includes the thermal cloud is to keep the dynamics of the condensate but assume that the thermal cloud always stays in **static thermal equilibrium**. While simple, this seems to often be a reasonable approximation.

We recall that, by definition, the time-independent **equilibrium form** of the distribution function is given by the requirement

$$C_{12}[f^{(0)}, \Phi^{(0)}] + C_{22}[f^{(0)}] = 0.$$

**Thermal equilibrium** is defined as the state where scattering **in** and **out** processes exactly balance, so that **both**  $C_{22} = 0$  and  $C_{12} = 0$ . This condition requires that

Comment on the **hydrodynamic** description of the condensate and thermal cloud:

The preceding analysis shows that the thermal cloud can be described using a **few local macroscopic variables** (like density  $\tilde{n}$ , pressure  $P$ , velocity  $v_n$ ) **only** under the condition that we have achieved **local equilibrium** through strong collisions. This is **collisional hydrodynamics**.

In the **collisionless region**, on the other hand, the full **atom distribution function  $f(p,r,t)$  is needed to describe the thermal cloud**. This is much more complicated.

In contrast, the “hydrodynamic” description of the condensate is **always valid** because all the atoms are in the same **single-particle quantum state**. This is specified by the condensate density  $n_c$  and velocity  $v_c$  (gradient of phase).

**C<sub>22</sub> only vanishes** if:

$$(1 + f_1)(1 + f_2)f_3f_4 - f_1f_2(1 + f_3)(1 + f_4) = 0,$$

**C<sub>12</sub> only vanishes** if:

$$[f_1(1 + f_2)(1 + f_3) - (1 + f_1)f_2f_3] = 0$$

Using the **Bose identity**  $f^0(x) \equiv \frac{1}{e^x - 1} = e^{-x}[1 + f^0(x)]$

one can verify that the **unique** solution of **C<sub>22</sub> = 0** is

$$f^0(\mathbf{p}, \mathbf{r}) = \frac{1}{e^{\beta_0[p^2/2m + U_0(\mathbf{r}) - \tilde{\mu}_0]} - 1}$$

The **additional** condition that **C<sub>12</sub> = 0** requires that the thermal cloud and condensate chemical potentials are **equal**.

This means that

$$\tilde{\mu}_0 = \mu_{c0} \equiv V_{trap}(\vec{r}) + gn_{c0}(\vec{r}) + 2g\tilde{n}_0(\vec{r})$$

We also note that since

$$U_0(r) - \tilde{\mu}_0 = gn_{c0}(\vec{r})$$

the **static equilibrium distribution** function is

$$f_i^0 = f^0(\mathbf{p}_i, \mathbf{r}) = \frac{1}{e^{\beta[p_i^2/2m + gn_{c0}(\mathbf{r})]} - 1}$$

However, we want to keep the dynamics of the condensate and as a result,  $C_{12}$  does **not vanish**. Calculation easily shows that

$$R_0(\mathbf{r}, t) = \frac{\hbar}{2\tau_{12}^c(\mathbf{r}, t)} \left( e^{\beta[\epsilon_c(\mathbf{r}, t) - \tilde{\mu}_0]} - 1 \right)$$

The **collision time** of a condensate atom with the thermal cloud atoms in equilibrium is

$$\frac{1}{\tau_{12}^c(\mathbf{r}, t)} \equiv \frac{2g^2}{(2\pi)^5 \hbar^7} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(\mathbf{p}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ \times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1}^0 - \tilde{\varepsilon}_{p_2}^0 - \tilde{\varepsilon}_{p_3}^0) (1 + f_1^0) f_2^0 f_3^0 .$$

where the condensate energy is

$$\varepsilon_c = \mu_c(\vec{r}, t) = V_{trap}(\vec{r}) + gn_c(\vec{r}, t) + 2gn_{c0}(\vec{r}) \\ = \mu_{c0} + \delta n_c(\vec{r}, t)$$

still involves the **time-dependent** condensate fluctuations. For small deviations  $\delta n_c$  from equilibrium, we can expand the exponential in  $R_0$  to obtain

$$\exp \beta[\mu_c(r, t) - \tilde{\mu}_0] - 1 \approx \beta g \delta n_c(r, t)$$



The end result is that in the **static thermal cloud approximation**, we have the GGP equation

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = \left[ -\frac{\hbar^2\nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) + gn_c(\mathbf{r},t) + 2g\tilde{n}_0(\mathbf{r}) - iR_0(\mathbf{r},t) \right]\Phi(\mathbf{r},t)$$

where

$$R_0(\vec{r},t) = \frac{\hbar\beta}{2\tau_{12}^0} g\delta n_c(\vec{r},t)$$

We see that we have derived a closed equation for condensate oscillations at finite temperature that now includes **damping from the coupling to a static thermal cloud**.

## Two-fluid collisional hydrodynamic region

I will now turn to the **opposite limit** where the interactions are **strong** enough to produce **local equilibrium** in the thermal gas.

In this limit, the thermal gas can be described by a few local variables, just as in **ordinary fluid dynamics**. **However, the new aspect is that these variables are coupled** to those describing the condensate degree of freedom. This leads to the **two-fluid hydrodynamics** first discussed for **superfluid Helium** by Landau (1941).

## How do we make the collision time small?

For a gas **above**  $T_{\text{BEC}}$ , collisions between the atoms in the thermal cloud of density  $\tilde{n}_0(r)$  give rise to the well known relaxation time:

$$\frac{1}{\tau_{\text{cl}}(\mathbf{r})} = \sqrt{2} \tilde{n}_0(\mathbf{r}) \sigma \bar{v}$$

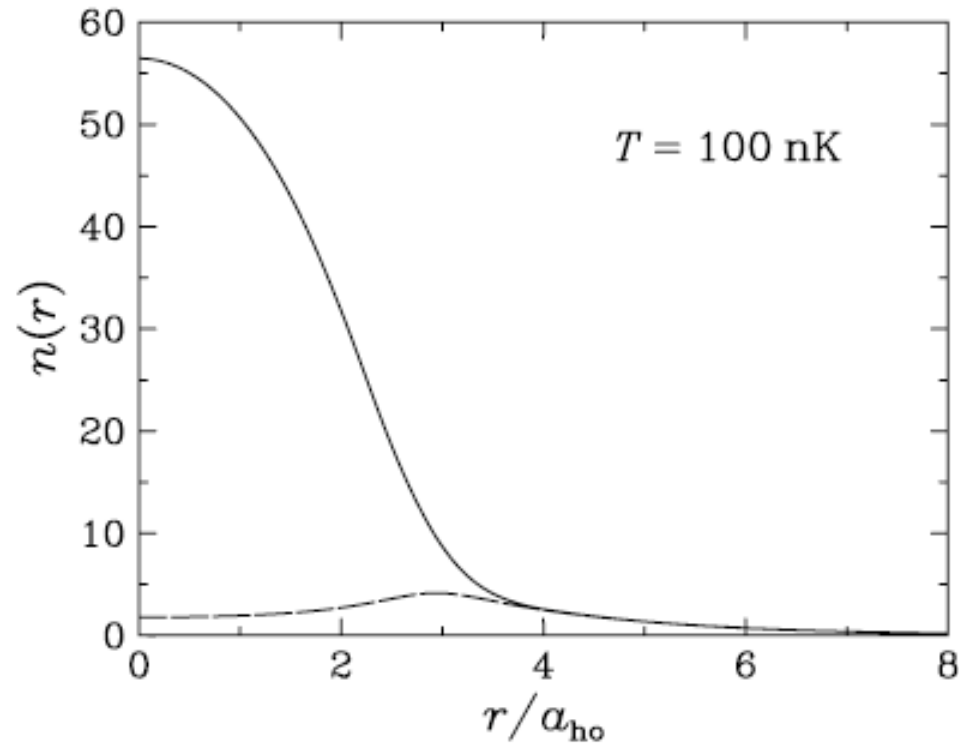
$\sigma = 8\pi a^2$ , where  $a$  is the s-wave scattering length for Bosons.

However, when a Bose condensate forms, it turns out that it is collisions between atoms in the **high density localized condensate** and the spread-out **low density thermal cloud** are most important. The result is:

$$\frac{1}{\tau_{\kappa}(\mathbf{r})} \simeq \sqrt{2} n_{c0}(\mathbf{r}) \sigma \bar{v}$$

**Need large value of the s-wave scattering length  $a$**

## Condensate vs thermal cloud density profiles in a harmonic trap: an example



This is for  $T \approx 0.75T_c$ . The high density of condensate atoms in the center of the trap potential means that the  **$C_{12}$  collisions always dominate over the  $C_{22}$  collisions.**

## Bose Distribution function in local equilibrium

$$C_{22}[f] = \frac{2g^2}{(2\pi)^5 \hbar^7} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4 \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \\ \times \delta(\tilde{\epsilon}_p + \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3} - \tilde{\epsilon}_{p_4}) [(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)]$$

For  $\tilde{f}(\mathbf{p}, \mathbf{r}, t) = \frac{1}{e^{\beta[\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2 + U - \tilde{\mu}]}} - 1$ , **Local equilibrium distribution**

$C_{22}[\tilde{f}, \Phi] = 0$  for **any** value of  $\Phi(\mathbf{r}, t)$

$$(1+f_1)(1+f_2)f_3f_4 - f_1f_2(1+f_3)(1+f_4) = 0$$

**All the quantities temperature, local velocity, chemical potential and the effective HF potential now depend on the position  $\mathbf{r}$  in the trap and the time  $t$ . This is what local hydrodynamic equilibrium means.**

We emphasize that this local equilibrium form is **imposed** by strong collisions such that  $C_{22} = 0$ , even though there is **no** explicit dependence on these collisions.

Using this local equilibrium distribution, we find

$$\begin{aligned} \Gamma_{12} = & -\frac{2g^2 n_c}{(2\pi)^5 \hbar^7} \left( 1 - e^{-\beta[\tilde{\mu} - \mu_c - \frac{1}{2}m(\mathbf{v}_n - \mathbf{v}_c)^2]} \right) \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \\ & \times \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta(\varepsilon_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) \\ & \times (1 + \tilde{f}_1) \tilde{f}_2 \tilde{f}_3. \end{aligned}$$

We note that this source term **only** vanishes if the term in the square bracket in the exponential **vanishes**.

We now **return to our kinetic equation** and take momentum moments, like we did earlier in deriving the continuity equation for the thermal cloud density.

$$\frac{\partial}{\partial t} \int \frac{d\vec{p}}{(2\pi\hbar)^3} f + \int \frac{d\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}}{m} \cdot \nabla_r f - \nabla_r U \cdot \int \frac{d\vec{p}}{(2\pi\hbar)^3} \nabla_p f = 0 + \int \frac{d\vec{p}}{(2\pi\hbar)^3} C_{12}$$

or

$$\frac{\partial \tilde{n}}{\partial t} + \nabla_r \cdot (\tilde{n} \vec{v}_n) - \nabla_r U \cdot \int \frac{d\vec{p}}{(2\pi\hbar)^3} \nabla_p f(\vec{r}, \vec{p}, t) = \Gamma_{12}(\vec{r}, t)$$



This integral vanishes

The thermal cloud local **density** and **velocity** variables are defined as

$$\tilde{n}(\mathbf{r}, t) \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f(\mathbf{p}, \mathbf{r}, t)$$

$$\mathbf{v}_n(\mathbf{r}, t) \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} \frac{f(\mathbf{p}, \mathbf{r}, t)}{\tilde{n}(\mathbf{r}, t)}$$

Let us write the kinetic equation in the **symbolic** form

$$\hat{L} \tilde{f} = C_{12}[\tilde{f}]$$

Multiplying by  $\mathbf{p}$  and integrate, after some work we find

$$\int d\mathbf{p} \mathbf{p} \left\{ \hat{\mathcal{L}} \tilde{f} = C_{12}[\tilde{f}] \right\} \rightarrow m \tilde{n} \left( \frac{\partial \mathbf{v}_n}{\partial t} + \frac{1}{2} \nabla v_n^2 \right) \\ = -\nabla \tilde{P}(\mathbf{r}, t) - \tilde{n} \nabla U(\mathbf{r}, t) - m(\mathbf{v}_n - \mathbf{v}_c) \Gamma_{12}[\tilde{f}],$$

For example, the first term in the kinetic equation gives

$$\frac{\partial(\tilde{n} \vec{v}_n)}{\partial t} = \tilde{n} \frac{\partial \vec{v}_n}{\partial t} + \vec{v}_n \frac{\partial \tilde{n}}{\partial t} = \tilde{n} \frac{\partial \vec{v}_n}{\partial t} + \vec{v}_n (-\nabla \cdot \tilde{n} \vec{v}_n + \Gamma_{12})$$

**This illustrates the manipulations that give the above result.**

The kinetic pressure term is defined as

$$\tilde{P}(\mathbf{r}, t) = \frac{m}{3} \int \frac{d\mathbf{p}}{(2\pi)^3} (\mathbf{p} - m\mathbf{v}_n)^2 \tilde{f}(\mathbf{p}, \mathbf{r}, t). \\ = \frac{1}{\beta \Lambda^3} g_{5/2}(z) \quad z(\mathbf{r}, t) \equiv e^{\beta(\mathbf{r}, t) [\tilde{\mu}(\mathbf{r}, t) - U(\mathbf{r}, t)]}$$



Finally, we multiply by  $p^2$  and integrate, giving

$$\begin{aligned} \int d\mathbf{p} p^2 \left\{ \mathcal{L} \tilde{f} = C_{12}[\tilde{f}] \right\} &\rightarrow \frac{\partial \tilde{P}}{\partial t} + \nabla \cdot (\tilde{P} \mathbf{v}_n) \\ &= -\frac{2}{3} \tilde{P} \nabla \cdot \mathbf{v}_n + \frac{2}{3} \left[ \mu_c + \frac{1}{2} m (\mathbf{v}_n - \mathbf{v}_c)^2 - U \right] \Gamma_{12}[\tilde{f}]. \end{aligned}$$

**In local equilibrium, the ZNG equations reduce to**

**CONDENSATE**

$$\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_c) = -\Gamma_{12}[f]$$

$$m \left( \frac{\partial}{\partial t} + \mathbf{v}_c \cdot \nabla \right) \mathbf{v}_c = -\nabla \mu_c,$$

**THERMAL CLOUD**

$$\frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\tilde{n} \mathbf{v}_n) = \Gamma_{12}^{(1)},$$

$$m \tilde{n} \left( \frac{\partial}{\partial t} + \mathbf{v}_n \cdot \nabla \right) \mathbf{v}_n = -\nabla \tilde{P} - \tilde{n} \nabla U - m(\mathbf{v}_n - \mathbf{v}_c) \Gamma_{12}^{(1)}$$

$$\frac{\partial \tilde{P}}{\partial t} + \nabla \cdot (\tilde{P} \mathbf{v}_n) = -\frac{2}{3} \tilde{P} \nabla \cdot \mathbf{v}_n + \frac{2}{3} \left[ \frac{1}{2} m (\mathbf{v}_n - \mathbf{v}_c)^2 + \mu_c - U \right] \Gamma_{12}^{(1)}$$

where one finds

$$\Gamma_{12}^{(1)}(\mathbf{r}, t) = \sigma_H \left\{ \nabla \cdot [n_c (\mathbf{v}_c - \mathbf{v}_n)] + \frac{1}{3} n_c \nabla \cdot \mathbf{v}_n \right\}$$

## Summary of what we have accomplished.

Assuming that the thermal cloud is described by the **local equilibrium Bose distribution**, we have derived a set of coupled hydrodynamic equations for both the condensate and the thermal cloud. These equations are a closed set for the **macroscopic variables** for both components. All reference to the thermal atom **distribution function** has disappeared. We are left with local densities, velocities, etc. that depend on  $\mathbf{r}$  and  $t$ .

We do not show any details, but one can prove that this set of coupled equations are **precisely equivalent** to the Landau two-fluid hydrodynamic equations introduced in Lecture 1. Landau formulated his theory using a **different** set of variables, namely **density and entropy fluctuations**. We need to go from our microscopic theory given in terms of the condensate and thermal cloud density fluctuations to the **thermodynamic** variables that Landau used.

## Landau two-fluid equations - from Lecture 1

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

$$m \frac{\partial \mathbf{j}}{\partial t} = -\nabla P - n \nabla V_{ex},$$

$$m \frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \mu,$$

$$\frac{\partial s}{\partial t} + \nabla \cdot (s \mathbf{v}_n) = 0$$

These equations do not look like the hydrodynamic equations we have derived. **But they can be proven to be equivalent!**

**Coupled equations for a uniform superfluid and normal fluid velocity potentials:**  $\delta\vec{v}_c = \nabla\phi_c$   $\delta\vec{v}_n = \nabla\phi_n$

$$m \frac{\partial^2 \phi_c}{\partial t^2} = gn_{c0} \nabla^2 \phi_c + 2g\tilde{n}_0 \nabla^2 \phi_n - g\delta\Gamma_{12}$$

$$m \frac{\partial^2 \phi_n}{\partial t^2} = \left( \frac{5\tilde{P}_0}{3\tilde{n}_0} + 2g\tilde{n}_0 \right) \nabla^2 \phi_n + 2gn_{c0} \nabla^2 \phi_c + \frac{2}{3}g\delta\Gamma_{12}$$

where

$$\delta\Gamma_{12} = \sigma_H n_{c0} \nabla^2 \left( \delta\phi_c - \frac{2}{3} \delta\phi_n \right)$$

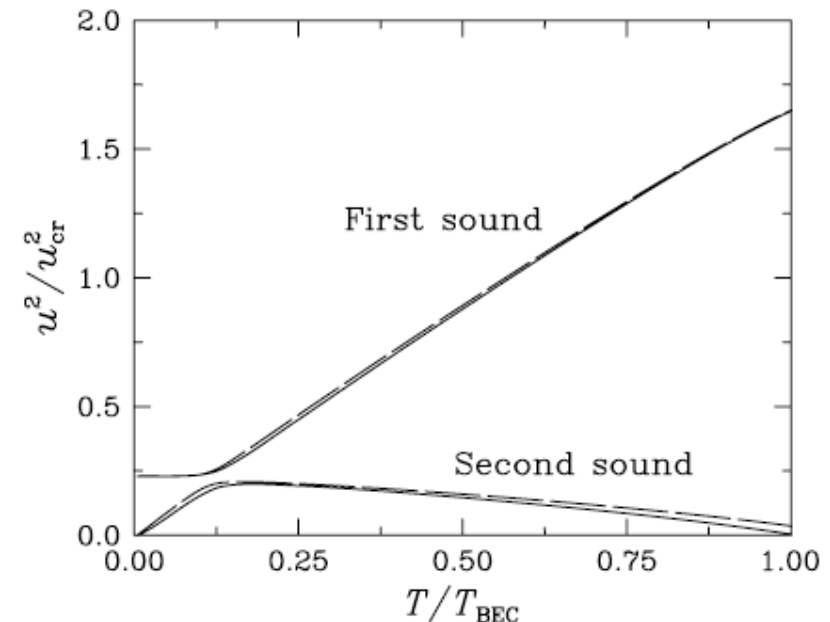
Assuming plane wave solutions, these two coupled equations reduce to two algebraic equations for the amplitudes  $\phi_c$  and  $\phi_n$  which can be easily solved for the **first and second sound velocities**. These results are **precisely** the same that we would get from the Landau equations.

## Hydrodynamic modes in dilute weakly interacting Bose gas -uniform gas

It turns out that the hydrodynamic normal modes in a dilute gas are largely **uncoupled oscillations** of the condensate and the thermal cloud components. These are the natural extensions of the pure condensate modes at  $T = 0$  and sound waves in a normal Bose gas **above**  $T_c$

$$u_1^2 = \frac{5}{3} \frac{\tilde{P}_0}{m\tilde{n}_0} + \frac{2g\tilde{n}_0}{m}$$
$$u_2^2 = \frac{gn_{c0}}{m}$$

These velocities only depend on the equilibrium pressure, thermal atom density and the condensate density, all at temperature  $T$ .



In our **trapped Bose gas**, of course, the elementary excitations that must be used to calculate the thermodynamic properties in the Landau two-fluid equations are not the phonon-roton excitations used for liquid  $^4\text{He}$ . These functions are those of a Bose gas of **atoms with the Hartree-Fock self-consistent energy spectrum**. In this **dilute** gas, the superfluid and normal fluid correspond to the condensate and the thermal cloud, respectively.

The Landau two fluid equations are now understood to be always correct for **superfluid hydrodynamics**, in liquid He, Bose gases and Fermi gases. This statement is equivalent to the statement that the equations of **classical fluid dynamics** are generic, valid for **all** normal fluids.

Our explicit **derivation** of Landau hydrodynamics is important conceptually since we started with a **Bose condensate**. It is the **basis of our whole discussion**. This is important since Landau in his derivation in 1941 did not explicitly use a Bose condensate, giving the **incorrect impression** that it was not essential. **It is!!**

Our derivation for a dilute Bose gas is also important since it gives one confidence that these two-fluid hydrodynamic equations are correct, even in **strongly interacting systems** where it is **difficult** to carry out an explicit proof. For example, they are expected to be valid in a Fermi gas superfluid when the interactions become **infinitely strong**.



## Can we observe first and second sound in trapped Bose gases?

It may be possible but it will be difficult because we cannot make the **s-wave interaction** large using a Feshbach resonance, because of large three-body losses which destroy the condensate. This might be overcome by doing the experiments very fast, before such processes can occur.

One possibility is to use **pulse propagation** along the axis of a long cigar-shaped trap, with a very weak axial trapping frequency. This makes it easier to satisfy the key condition

$$\tau \ll T \Rightarrow \omega\tau \ll 1$$

with the collective mode frequency  $\omega$  being of the order of the small axial trap frequency  $\omega_0$ .

A much more promising system is a **Fermi superfluid gas**.