The dynamics of ultracold superfluid gases at finite temperatures .

Allan Griffin, *University of Toronto, Canada*

Lecture 1: A very long introduction

1. Review the collective oscillations of a pure Bose condensate at T = 0 in a trapped atomic gas using the GP equation of motion.

2. **Pose the question** of generalizing this to **finite temperature**, when we have to deal with a condensate **coupled** to a thermal cloud - what replaces the GP equation when we have **two fluids**?

3. Review the physics of the **Landau two-fluid hydrodynamic** equations which describe superfluid ^{4}He and second sound.

4. **Pose the question**: Under what conditions are the Landau two-fluid equations valid in **Bose** and **Fermi** superfluid atomic gases.

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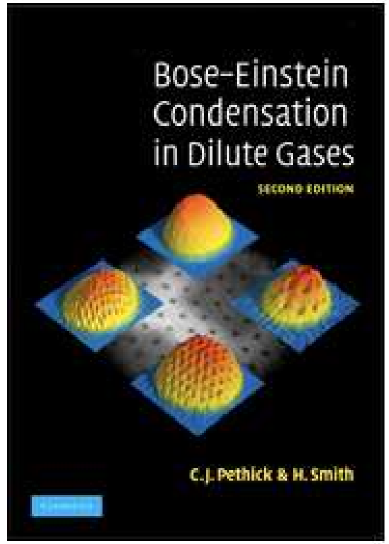
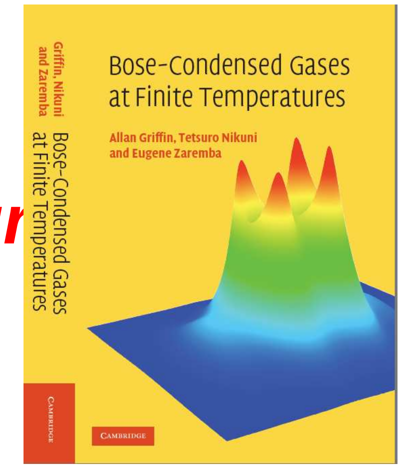
My main collaborators

Edward Taylor (University of Trento, Italy and Ohio State, USA)
Eugene Zaremba (Queen's University, Canada)
Tetsuro Nikuni (Tokyo University of Science, Japan)



C. J. Pethick and H. Smith,
Bose-Einstein Condensation in Dilute Gases
second edition
(Cambridge , 2008**)**



A. Griffin, T. Nikuni and E. Zaremba,
Bose-Condensed Gases at Finite Temperatur
(Cambridge, 2009**)**

I will refer to this book (GNZ) for details

Why do we need low temperatures for superfluids?

- Quantum effects are **smeared out at high temperatures** due
to thermal motion. This is why physicists have been on a long quest to go to lower and lower temperatures. Life is more **interesting** as $T \to 0$, where more delicate phases of matter can become **stabilized**. **Superfluidity only lives at low T.**

- BEC and ultracold atomic gases are just the latest spectacular discovery in this **quest for absolute zero** over the last 100 years.

What do we mean by low temperatures?

milli **micro**

nano

Quantum statistics

- Two classes of particles in nature: ***Bosons*** and ***Fermions***
- ***Bosons***: integral spin particles – photons, mesons, **atoms** with an even number of neutrons(^{4}He, ^{87}Rb, ^{7}Li ,^{23}Na,…)
- ***Fermions***: - integral spin particles – electrons, protons, neutrons, **atoms** with an odd number of neutrons (^{3}He,^{6}Li, ^{40}K,..)

What is the origin of the physical difference between Fermions and Bosons?

At the microscopic level, it is associated with the behaviour of the many-body wave function under the exchange of ***identical*** *particles:*

$|\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)|^2$ - probability of finding 1 at $\mathbf{r}_1$, 2 at $\mathbf{r}_2$, …

$|\Psi(\mathbf{r}_2, \mathbf{r}_1, ..., \mathbf{r}_N)|^2$ - probability of finding 1 at $\mathbf{r}_2$, 2 at $\mathbf{r}_1$, …

Particles are ***indistinguishable***, these two probabilities must be the ***same***, and hence we must have:

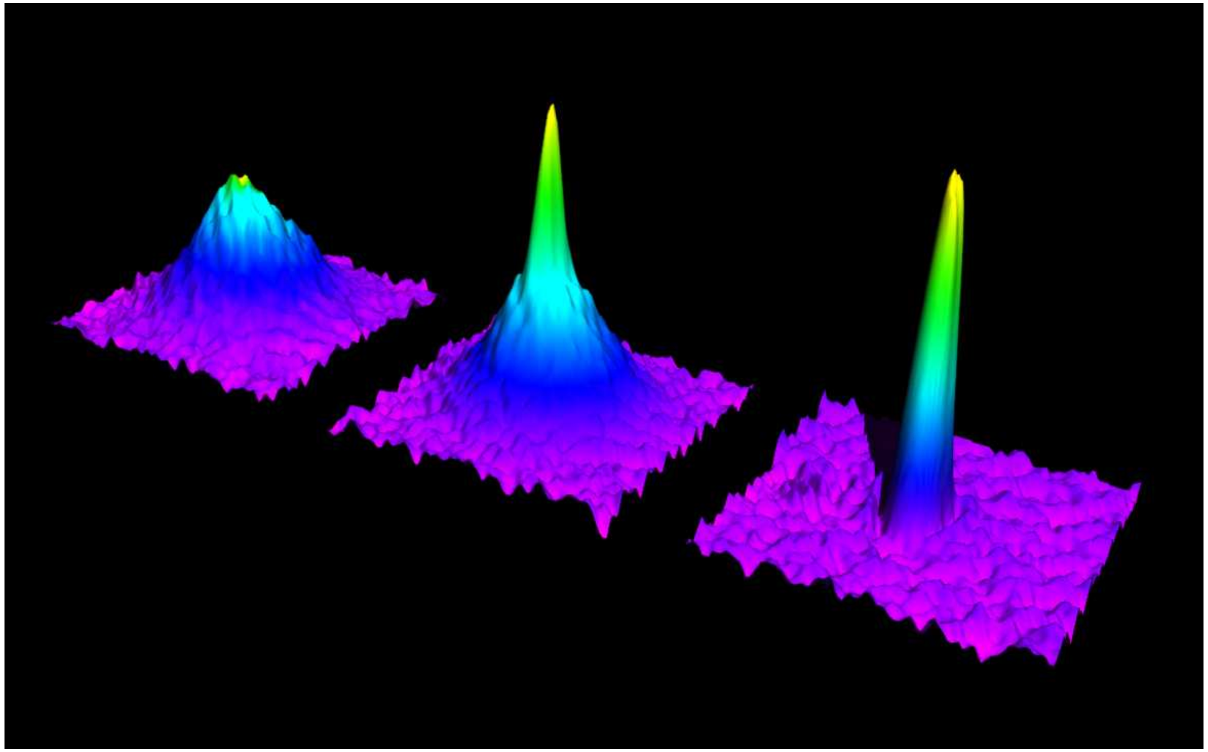
$$\Psi(\mathbf{r}_2, \mathbf{r}_1, ..., \mathbf{r}_N) = \pm\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)$$

+ sign: Bosons

– sign: Fermions

The density of Bosons in a trap as the temperature goes from above T_{BEC} to below

$T > T_c$ $T < T_c$ $T \ll T_c$



The peak on the **right** is almost a **pure atom condensate.** (from *MPI, Munich*)

One sees the characteristic **bi-modal density profile**, with a broad **thermal cloud of non-condensate atoms** and a sharp **condensate peak. In my lectures, I will be talking about superfluid at finite temperatures, ie, the middle peak.**

Dynamics of a pure condensate at T = 0

See Ch.2 of GNZ

In the last decade, there have been extensive studies of atomic Bose gases when all the atoms are in the same single particle quantum state described by the **Gross-Pitaevskii time-dependent equation** that Franco has discussed:

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{trap}}(\mathbf{r}) + g|\Phi(\mathbf{r},t)|^2\right]\Phi(\mathbf{r},t)$$

The equilibrium value of the condensate wave function $\Phi_0(\mathrm{r})$

$$\mu_{c0}\Phi_0(\mathbf{r}) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{trap}}(\mathbf{r}) + g|\Phi_0(\mathbf{r})|^2\right]\Phi_0(\mathbf{r})$$

In the **Thomas Fermi (TF) approximation**, the kinetic energy can be
neglected for a large condensate. This gives the equilibrium condensate density pr ... ical potential

$$n_{c0}(r) = |\Phi_0|^2 = \frac{1}{g}\left[\mu_{c0} - \tfrac{1}{2}m\omega_0^2 r^2\right] \geq 0$$

The GP equation in terms of density and velocity

It is much more physical to write the GP equation in terms of density and phase variables

$$\Phi(\mathbf{r},t) = \sqrt{n_c(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)}$$

Substituting this into the GP equation, the left had side is

$$\frac{i\hbar}{\Phi}\frac{\partial\Phi}{\partial t} = -\hbar\frac{\partial\theta}{\partial t} + \frac{i\hbar}{2n_c}\frac{\partial n_c}{\partial t}$$

The right hand side reduces to

$$\begin{aligned}\frac{1}{\Phi}\left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{trap}} + gn_c\right)\Phi = &- \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}} + \frac{\hbar^2}{2m}|\nabla\theta|^2 + V_{\text{trap}} + gn_c \\ &- \frac{i\hbar^2}{2mn_c}\left(\nabla n_c\cdot\nabla\theta + n_c\nabla^2\theta\right).\end{aligned}$$

Separating out the real and imaginary parts gives two equations for the phase and ndensity

The real parts gives an equation for the **condensate velocity**

$$m\frac{\partial \mathbf{v}_c}{\partial t} = -\nabla \varepsilon_c$$

where the local energy of a condensate atom is $\varepsilon_c \equiv \mu_c + \frac{1}{2}mv_c^2$
and generalized condensate chemical potential is

$$\mu_c(\mathbf{r},t) \equiv -\frac{\hbar^2\nabla^2\sqrt{n_c(\mathbf{r},t)}}{2m\sqrt{n_c(\mathbf{r},t)}} + V_{\text{trap}}(\mathbf{r}) + gn_c(\mathbf{r},t)$$

Here we introduced the condensate velocity $\mathbf{v}_c(\mathbf{r},t) \equiv \frac{\hbar}{m}\nabla\theta(\mathbf{r},t)$

The imaginary parts give the a **density conservation** equation

$$\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_c) = 0,$$

We have thus reduced the GP equation to two separate coupled equations for the condensate density and velocity variables.

The GP equation is only a simple mean field for the condensate but the concept on a macroscopic wavefunction is valid in all Bose superfluids, even at finite temperatures and strong interactions. It is the basis for understanding of superfluid ^{4}He. This means that the superfluid velocity is always related to the gradient of the phase.

$\Phi(r, t) = (n_c)^{1/2} e^{i\theta}$: amplitude and phase variables

The condensate **density** is $n_c(r, t) = |\Phi(r, t)|^2$

The superfluid **velocity** is $v_c(r,t) = \frac{\hbar}{m}\nabla\theta(r,t)$.

It is the **phase** of $\Phi(r,t)$ which is the **origin** of all the wavelike and superfluid properties of the condensate.

The GP equation has been reduced to **coupled equations for two local variables**, the density and velocity, which depend on **r** and *t*. This is because all the atoms are in the **same** single particle quantum state.

This description is similar to **ordinary fluid dynamics** which describes normal fluids. Hydrodynamics involves a few differential equations for local macroscopic variables such as the local density, velocity, pressure, temperature, etc There is **no reference** to the underlying millions of atoms moving in the fluid, all with different velocities.

As a result, the GP theory written in terms of condensate the density and velocity is called the **hydrodynamic theory of superfluids**. However, this language is misleading since hydrodynamics of a normal fluid is **only valid under certain conditions** which lead to something called "local equilbrium". We will discuss this in great detail when we consider **thermal cloud dynamics** in the next lecture.

We can **linearize** these coupled equations around the equilbrium value $n_{c0}(r)$ and set $v_{c0}(r) = 0$. Using the fact that μ_{c0} is independent of position in the trap, we obtain the **coupled equations**:

$$\frac{\partial \delta n_c}{\partial t} = -\nabla \cdot (n_{c0}(\mathbf{r})\delta \mathbf{v}_c)$$

$$\frac{\partial \delta \mathbf{v}_c}{\partial t} = -\frac{g}{m}\nabla \delta n_c.$$

Combining these, we find the **wave equation**:

$$\boxed{\frac{\partial^2 \delta n_c(\mathbf{r},t)}{\partial t^2} = \frac{g}{m}\nabla \cdot [n_{c0}(\mathbf{r})\nabla \delta n_c(\mathbf{r},t)]}$$

This closed equation for condensate **density fluctuations** was first derived by **Stringari** in 1996 and has been very useful in the study of the collective modes in a pure condensate at $T = 0$. Note that it could equally be written in terms of the **velocity fluctuations** related to the gradient of the phase.

As a first application of the Stringari wave equation, let us use it for a **uniform Bose gas** (no trap potential). In this case, the normal solutions are simple plane waves

$$\delta n_c(\mathbf{q}, t) = \delta n_{q\omega} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$$

Substituting into the Stringari equation, we obtain

$$-\omega^2 \delta n_{\mathbf{q}\omega} = \frac{g n_{c0}}{m}(-q^2)\delta n_{\mathbf{q}\omega}.$$

with the solutıon

$$\omega^2 = c_0^2 q^2, \text{ where } c_0 = \sqrt{g n_{c0}/m}.$$

This corresponds to the **Bogoliubov phonon mode** (1947). One can show that all Bose superfluids , even at finite T, have such a **Goldstone phonon** mode. Omitting the TF approximation, we obtain full Bogoliubov **excitation** spectrum at T = 0 :

$$\hbar\omega = \left(\varepsilon_q^2 + 2 g n_{c0}\varepsilon_q\right)^{1/2} \approx \varepsilon_q + g n_{c0} \qquad \text{, for large q}$$

A condensate collective mode in a trap: breathing mode

Ansatz: $$\delta \mathbf{v}_c(\mathbf{r},t) = A\mathbf{r}e^{-i\omega t}, \quad r \le R_{TF}$$

The continuity equation gives:

$$\begin{aligned} -i\omega\delta n_c &= -\nabla\cdot(n_{c0}(\mathbf{r})\delta\mathbf{v}_c) \\ &= -\frac{\mu_{c0}}{g}A\left(3 - 5\frac{r^2}{R_{TF}^2}\right)e^{-i\omega t} \end{aligned}$$

Inserting this into the Stringari wave equation gives:

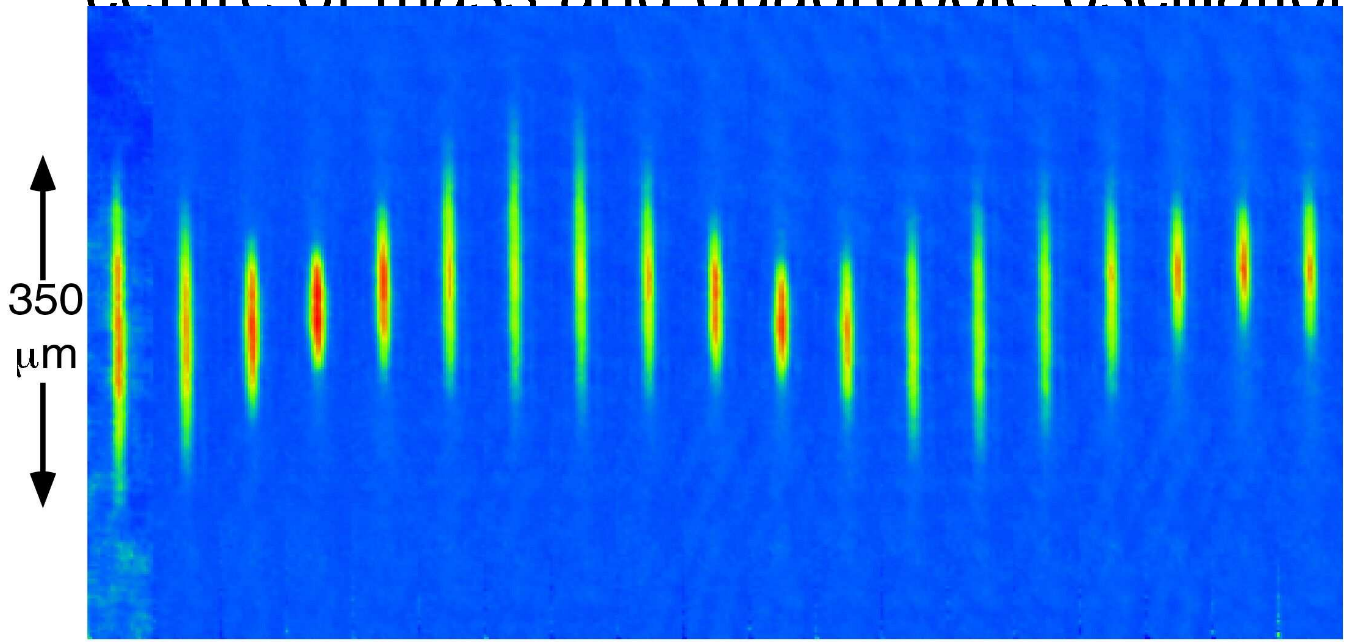
$$\begin{aligned} -\omega^2\delta n_\omega(\mathbf{r}) &= -\frac{\mu_{c0}}{m}\nabla\cdot\left[\left(1 - \frac{r^2}{R_{TF}^2}\right)B\frac{5}{3}\frac{2\mathbf{r}}{R_{TF}^2}\right] \\ &= -5\omega_0^2\delta n_\omega(\mathbf{r}). \end{aligned}$$

Thus this lowest frequency **breathing mode** in a trap has a frequency

$$\omega = \sqrt{5}\omega_0$$

Dynamics of the macroscopic wavefunction

Solutions of the GP equation describe the behavior of $\Phi(r, t)$. In equilibrium, $\Phi_0(r)$ is time-independent. One can induce changes from equilibrium using external **laser fields** and excite **collective oscillations** of the condensate. These are like phonons in a crystal. Here is a centre of mass and quadrupole oscillation of ^{23}Na atoms



5 milliseconds per frame

Two distinct oscillations Are visible

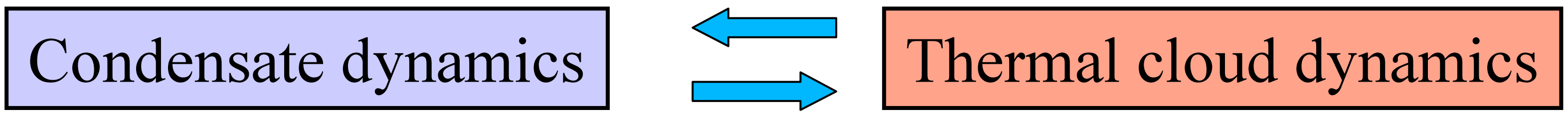
From Ketterle MIT group

Dynamics of a Bose condensate coupled to a thermal cloud

I now want to introduce a **major theme** of these lectures, which is what happens to the T = 0 GP theory when we deal with a Bose-condensed gas at **finite temperatures.** The details will come in my second lecture.

Here I want to tell you why this **extension** is so important. It will give us a deeper understanding of a Bose gas superfluid. In particular, it will relate this trapped gas to superfluid ^{4}He.

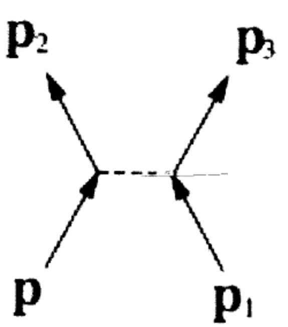
What can we expect this new theory to involve? Clearly we will now have two coupled components:



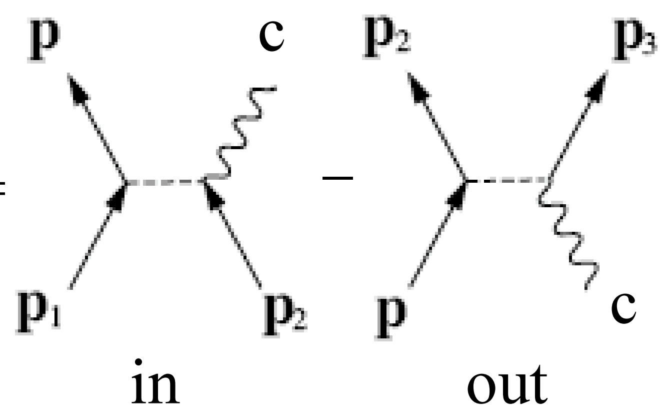
Both systems will influence each other:

- By **self-consistent mean fields** produced by the condensate and by the thermal atoms act on one another and themselves.
- **Collisions** between atoms in each part,which can transfer atoms between the condensate and the thermal cloud.

$C_{22} =$



$C_{12} =$



C_{22} collisions between atoms in the thermal cloud- as in a normal Bose gas.

These C_{12} collisions change the number of atoms in the condensate. Atoms scatter into or out of condensate.

$p_1 + p \Rightarrow p_2 + p_3$

How do we describe the thermal cloud atoms? Clearly their dynamics will be governed by some sort of **Boltzmann or kinetic equation -** as in a classical gas.

In our work, we use the simplest kind of Boltzmann equation for a semi-classical **single particle distribution function**:

$$f(\boldsymbol{p}, \boldsymbol{r}, t)$$

This distribution tells us, in a non-equilibrium state, how many atoms have **momentum** $\boldsymbol{p}$, at **position** $\boldsymbol{r}$ and at **time** t. Collisions with other atoms in the cloud and in the condensate bring this distribution back to the thermal equilbrium value $f^0(\boldsymbol{p}, \boldsymbol{r})$, which is **time-independent** but still depends on the position.

This **semiclassical** approximation is okay at higher tempertures, where the thermal cloud can be treated as atoms moving in the trap potential and in **self-consistent HF fields.**

The collisions described by C_{22} and C_{12} will take into account **Bose statistics**. This is very important.

The coupled generalized GP equation for the condensate atoms and the Boltzmann kinetic equation for the thermal cloud atom distribution function will allow us to work out the complete **non-equilibrium behavior of a trapped Bose-condensed gas.**

There are two regions of interest at finite T:

Collisionless region - where mean fields mainly determine the solution of the kinetic equation.

Collisional hydrodynamic region - where atomic collisions determine the **form** of the solution of the kinetic equation.

Most experiments on collective modes in trapped Bose gases **so far** have studied have worked in the **collisionless limit,** defined by

$$\tau >> T \Rightarrow \omega\tau >> 1$$

This means that the collisional mean free path is much larger than the wavelength of the the collective mode. Collisions are present , but do not lead to **local equilibrium.**

The **GP theory** at T = 0 is a pure **collisionless approximation**. It only includes the **self-consistent mean field** of the condensate atoms.

The hydrodynamic collision-dominated region describes **collective oscillations** with a period T only if the appropriate atomic collision time τ satisfies the condition

$$\tau << T \Rightarrow \omega\tau << 1$$

This is equivalent to saying that the mean free path of the atoms must be**much smaller** than the wavelength of the collective mode

In a **normal fluid** , hydrodynamics allows one to describe the dynamics in terms of a few macroscopic local variables, like **local** density $n(\mathbf{r},t)$,pressure $P(\mathbf{r}, t)$, temperature $T(\mathbf{r}, t)$, etc. This requires **local equilibrium**.

We finally come to the topic we are interested in: namely the dynamics of a trapped gas when both the condensate and thermal cloud have a hydrodynamic description in terms of a few variables like density, velocity, etc

Two-fluid hydrodynamics

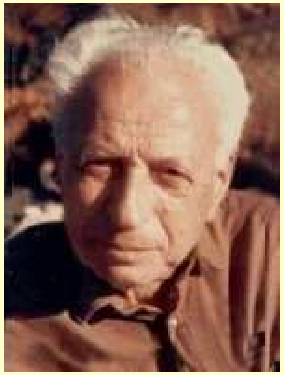
We are interested in this **two-fluid description** of the condensate and thermal cloud in Bose gases.

Before discussing this region in dilute Bose gases, we want to **review the standard description that is used for superfluid ^{4}He**. Two-fluid hydrodynamics was first developed for this system by Landau in 1941.

It is now realized that his set of equations is valid for the low frequency dynamics of **all** superfluids at finite temperatures which involve a superfluid and normal fluid coupled together.

We will show in the next lectures that it indeed is correct for both **Bose and Fermi superfluid gases** if local equilibrium can be achieved .

Tisza-Landau two-fluid hydrodynamics 1938-1941



Tisza

Landau

***Superfluid*: *component of liquid which is associated with macroscopic occupation (BEC) of **one** **single-particle** state .Carries zero entropy, flows without dissipation with an irrotational velocity.*

Normal fluid: *comprised of **incoherent** thermal excitations , behaves like any fluid at finite temperatures in **local thermodynamic** equilibrium. This requires strong collisions.*

Laszlo Tisza died on April 15, 2009. He was 101.

Landau two-fluid hydrodynamic equations (1941)

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

⇦ **Conservation** equation

$$m\frac{\partial \mathbf{j}}{\partial t} = -\nabla P - n\nabla V_{\mathrm{trap}},$$

⇦ **Euler equation**

$$m\frac{\partial \mathbf{v}_{\mathrm{s}}}{\partial t} = -\nabla \mu,$$

⇦ **This says superfluid is irrotational.**

$$\frac{\partial s}{\partial t} + \nabla \cdot (s\mathbf{v}_{\mathrm{n}}) = 0$$

⇦ **This says the superfluid carries no entropy.**

The mass density and current are the sum of the superfluid and normal fluid components:

$$mn \equiv \rho = \rho_{\mathrm{s}} + \rho_{\mathrm{n}},$$

$$m\mathbf{j} \equiv \rho_{\mathrm{s}}\mathbf{v}_{\mathrm{s}} + \rho_{\mathrm{n}}\mathbf{v}_{\mathrm{n}}$$

These equations can be written in different forms.

Landau two-fluid equations of motion (non-dissipative limit) in linearized form:

$$\frac{\partial \delta n}{\partial t} + \boldsymbol{\nabla} \cdot \delta \mathbf{j} = 0$$

$$m\frac{\partial \delta \mathbf{j}}{\partial t} = -\boldsymbol{\nabla}\delta P - \delta n \boldsymbol{\nabla} U_{ext}$$

$$m\frac{\partial \delta \mathbf{v}_s}{\partial t} = -\boldsymbol{\nabla}\delta \mu$$

All local variables are linearized around their equilibrium values.

$$\frac{\partial \delta s}{\partial t} + \boldsymbol{\nabla} \cdot (s_0 \delta \mathbf{v}_n) = 0\,,$$

The linearized current and density fluctuations are

$$m\delta \mathbf{j}(\mathbf{r},t) = \rho_{s0}(\mathbf{r})\delta \mathbf{v}_s(\mathbf{r},t) + \rho_{n0}(\mathbf{r})\delta \mathbf{v}_n(\mathbf{r},t),$$
$$m\delta n(\mathbf{r},t) = \delta \rho_s(\mathbf{r},t) + \delta \rho_n(\mathbf{r},t).$$

The T=0 limit of the Landau hydrodynamic equations describes a **pure superfluid** since:

$$T = 0 \Rightarrow \rho_{n0} = 0 \Rightarrow \rho_{s0} = \rho_0 = mn_0$$

$$\frac{\partial^2 \delta\rho(r,t)}{\partial t^2} - \nabla \cdot \left[n_0(r) \nabla \delta\mu(r,t) \right] = 0$$

$$\delta\mu(r,t) = \frac{\partial \mu_0(n)}{\partial n} \frac{\delta\rho(r,t)}{m}$$

This reduces the problem to finding the **equilibrium equation of state**, μ as a function of the density **n**.
It describes a one-component **pure superfluid** at T = 0.
It reduces to the **Stringari wave equation** at T = 0 in the GP mean field approximation.

Landau's two-fluid hydrodynamic equations describes the fluctuations of a Bose condensate in local equilibrium with the normal fluid, brought about by strong collisions . It is an **extension of ordinary fluid dynamics** described in terms of the usual hydrodynamic variables , now including the superfluid degree of freedom.The equations describe the dynamics of a superfluid coupled to a normal fluid at finite T, which has been brought into **local hydrodynamic equilibrium** by rapid collisions.

Landau's two-fluid equations describe the **low frequency dynamics** of superfluid ^{4}He, *s*-wave BCS superconductors, as well as superfluid Bose and Fermi gases. These equations involve the superfluid and normal fluid densities and equilibrium thermodynamic functions, but make no explicit reference to atoms or inter-atomic interactions.

What are conditions for the Landau equations?

We need the **normal fluid** dynamics to be described in terms of a few macroscopic local variables, like **local** pressure $P(\mathbf{r}, t)$, **local** temperature $T(\mathbf{r}, t)$, etc. This requires **local equilibrium**. Just as in normal fluids, this in turn requires strong collisions between the atoms.

The two-fluid equations will describe **collective oscillations** with a period T only if the appropriate **atomic collision time τ** satisfies the condition

$$\tau << T \Rightarrow \omega\tau << 1$$

This is equivalent to saying that local equilibrium requires the mean free path of the atoms to be **much smaller** than the wavelength of the collective mode.

The structure of the Landau equations is universal.The only difference between superfluid ^{4}He, a superfluid Bose gas and a superfluid Fermi gas is in the evaluation of the local thermodynamic functions, such as the pressure, entropy, superfluid density etc. Only these require a microscopic theory for the thermal excitations of the particular system of interest.

The **corrections** to these equations give hydrodynamic damping involving transport coefficients .These are proportional to the various **transport collision times** τ, which are **small** for **strong interactions** between atoms. Two-fluid hydrodynamics **requires strong interactions.**

How do we make the collision time small?

As an example, above $T_{BEC,}$, collisions between the atoms in the thermal cloud of density $\tilde{n}_0(r)$ give rise to the classical gas collision time:

$$\frac{1}{\tau_{cl}(\mathbf{r})} = \sqrt{2}\tilde{n}_0(\mathbf{r})\sigma\bar{v},$$

σ **= 8π*a*²**, where *a* is the s-wave scattering length.

Things improve when a Bose condensate forms, since the collisions between atoms in the **high density localized condensate** and the spread-out **low density thermal cloud** are most important. The new collision time is given by

$$\frac{1}{\tau_{\kappa}(\mathbf{r})} \simeq \sqrt{2}n_{c0}(\mathbf{r})\sigma\bar{v}$$

We need high density or a large scattering length *a* to reach the conditions for **collisional hydrodynamics.**

Using the thermodynamic identity,

$$n_0 \delta\mu = -s_0 \delta T + \delta P,$$

one can reduce the two-fluid equations to **two coupled wave equations**

$$\frac{\partial^2 \delta\rho}{\partial t^2} = -m\nabla \cdot \frac{\partial \delta \mathbf{j}}{\partial t} = \nabla^2 \delta P.$$

$$\frac{\partial^2 \delta \bar{s}}{\partial t^2} = \bar{s}_0^2 \left(\frac{\rho_{s0}}{\rho_{n0}}\right) \nabla^2 \delta T$$

where the entropy per unit mass is defined as $\bar{s} \equiv s/\rho$

We can **eliminate** the fluctuations in pressure and temperature using the relations

$$\delta P = \left(\frac{\partial P}{\partial \rho}\right)_{\bar{s}} \delta\rho + \left(\frac{\partial P}{\partial \bar{s}}\right)_{\rho} \delta\bar{s},$$

$$\delta T = \left(\frac{\partial T}{\partial \rho}\right)_{\bar{s}} \delta\rho + \left(\frac{\partial T}{\partial \bar{s}}\right)_{\rho} \delta\bar{s}\,.$$

We see that the coefficients in these coupled equations involve a lot of **thermodynamic functions and derivatives**. These must be calculated and are **different** for each superfluid

First and second sound in uniform superfluids

In **uniform** Bose superfluids, we know that the solutions of the coupled wave equations are plane waves

$$\delta\rho, \delta\bar{s} \propto e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$$

Substituting these into the wave equations, one finds that they reduce to two coupled **algebraic equations** with phonon or **sound wave** solutions

$$\omega^2 = u^2 q^2$$

The two values of the sound velocity are the solutions of the quadratic equation for u^2 (Landau,1941)

$$u^4 - u^2\left[\left(\frac{\partial P}{\partial \rho}\right)_{\bar{s}} + \frac{\rho_{s0}}{\rho_{n0}}\frac{T\bar{s}_0^2}{\bar{c}_v}\right] + \frac{\rho_{s0}}{\rho_{n0}}\left(\frac{T\bar{s}_0^2}{\bar{c}_v}\right)\left(\frac{\partial P}{\partial \rho}\right)_T = 0$$

This equation for the velocities of first and second sound waves is **very general**. It applies to liquid Helium as well as to uniform **Bose and Fermi superfluid gases** , as long as the conditions for the Landau two-fluid equations are satisfied.

It turns out that in liquid ^{4}He, the solutions simplify because the pressure and temperature are very **weakly coupled**, so that

$$\left(\frac{\partial P}{\partial T}\right)_\rho = 0$$

In this case, the first and second sound velocities are:

$$\boxed{c_1^2 = \left(\frac{\partial P}{\partial \rho}\right)_{\bar{s}}, \quad c_2^2 = T\frac{\bar{s}_0^2}{\bar{c}_v}\frac{\rho_{s0}}{\rho_{n0}}}$$

One can insert these solutions into the Landau equations and check:

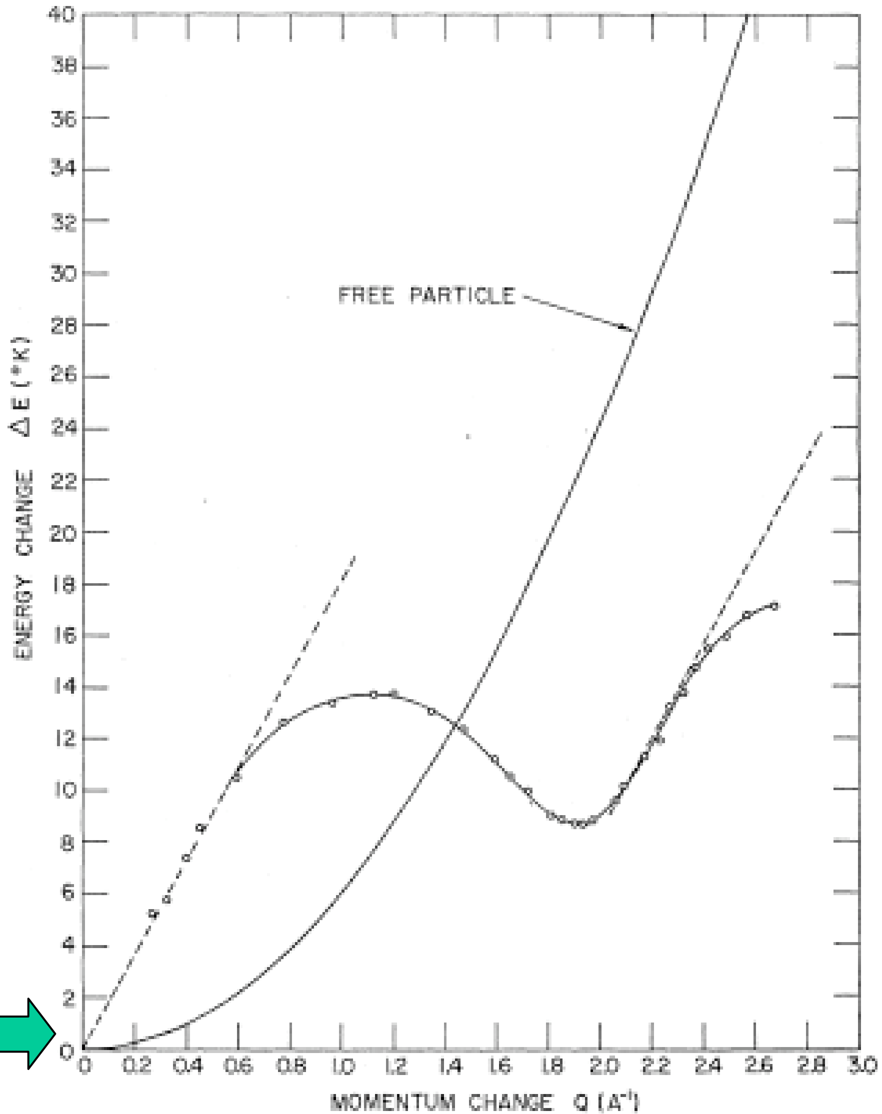
First sound is a simple in-phase motion $\delta \mathbf{v}_\mathrm{n} = \delta \mathbf{v}_\mathrm{s}$.

Second sound is an out-of -phase motion $\rho_\mathrm{n0}\, \delta \mathbf{v}_\mathrm{n} = -\rho_\mathrm{s0}\, \delta \mathbf{v}_\mathrm{s}$.

Elementary excitations in superfluid ^{4}He
from neutron scattering data at T = 1K

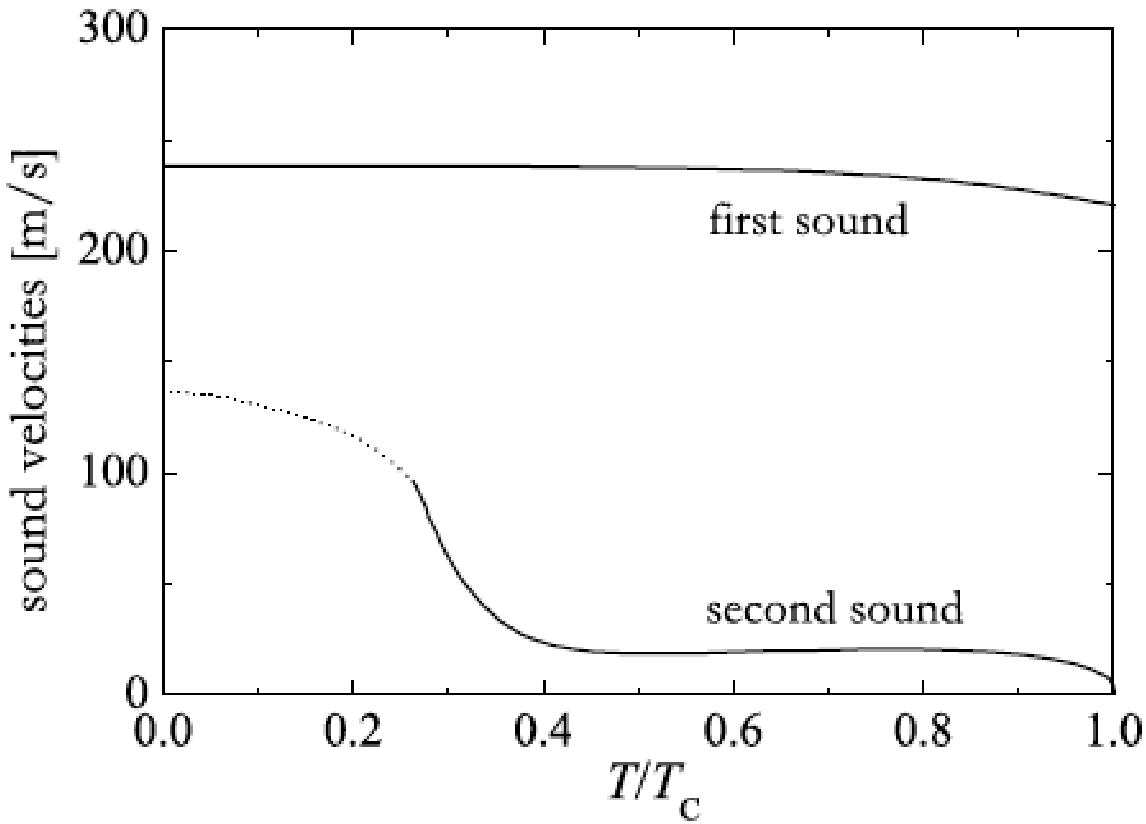
Henshaw &Woods, 1961

rve for liquid helium at 1.12°K under its normal vapor pressure. The parabolic curve rising from the origin represents the theoretically calculated dispersion curve for free helium atoms at absolute zero. The open circles correspond to the energy and momentum of the measured excitations. A smooth curve has been drawn through the points. The broken curve rising linearly from the origin is the theoretical phonon branch calculated from a velocity of sound of 237 meters sec^{-1}. The dotted curve drawn through the point at 2.27 A^{-1} has been drawn with a slope equal to the velocity of sound.



First and second sound only appear down here

First and second sound in superfluid ^{4}He



The detection of second sound by Peshkov in Moscow in 1944 was one of **greatest discoveries** in low temperature physics.
It would be great to find **second sound in superfluid gases!**

Second sound in superfluid atomic gases??

We now come **back to atomic gases** and ask, can we achieve the conditions that validate the Landau two-fluid description?

It looks like it will be **difficult to do** this in atomic Bose gas. This is because one cannot make the interactions strong enough or the density to satisfy the **local equilbrium** criterion.

However, **second sound** in Fermi superfluid gases seem much more feasible.This is because we can use a **Feshbach resonance** to adjust the size of the *s*-wave scattering length between two Fermi atoms in different hyperfine states.This allows us to study the famous **BCS-BEC crossover** in Fermi gases. I will review this topic in my third lecture.

The bound states in interacting Fermi gases are bosonic in nature and hence can **Bose condense**, just like Bose atoms can. As a result, in a discussion of trapped Fermi gases, a Bose condensate will appea once again, except that it now describes a molecular Bose condensate

A strongly interacting superfluid Fermi gas with a Feshbach resonance give us the **perfect system** to reach two-fluid hydrodynamics in gases the precise analogue of what Landau studied in 1941 in liquid Helium.

The main problem is that the **thermodynamics at finite temperatures** in the **BCS-BEC crossover** is quite complicated, since it involves both Fermi and Bose elementary excitations. Remember, the two-fluid equations have coefficients which involve quantities like entropy, superfluid density, etc.

This concludes the long overview of my lectures!!

I have tried to motivate **why** we are interested in studying superfluidity in atomic gases at **finite temperatures**. It makes **connections** between superfluid **gases** and superfluid liquids.

It allows to understand superfluidity more deeply when the quantum **superfluid component is coupled to a normal fluid**, yet the combined system **still** exhibits superfluidity.

The two-fluid collision dominated hydrodynamic region is the **most interesting** extension from simple GP theory at T = 0, because **both the condensate and the non-condensate** components are individually described by a few "hydrodynamic"variables, like density and velocity.

Finally, two-fluid hydrodynamics is by definition **correct in the limit of strong interactions**, which can be studied at **unitarity** in the BCS-BEC crossover region. This makes connection with all sorts of exotic systems, like quark-gluon plasmas.