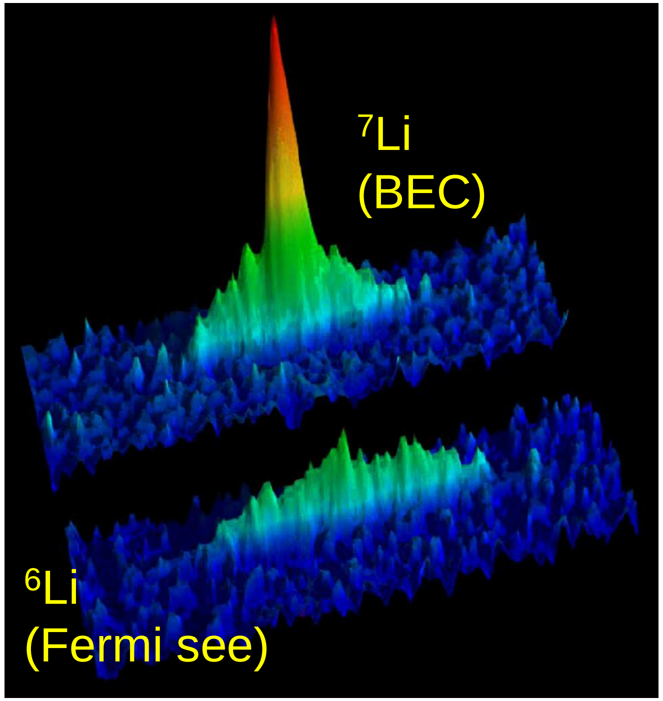
Fermions

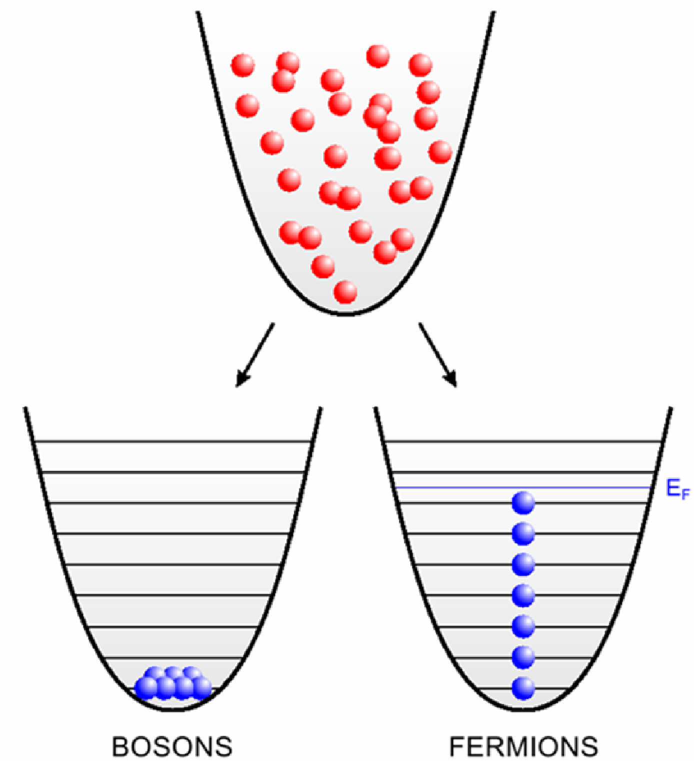
Ultracold fermions

A simple argument:

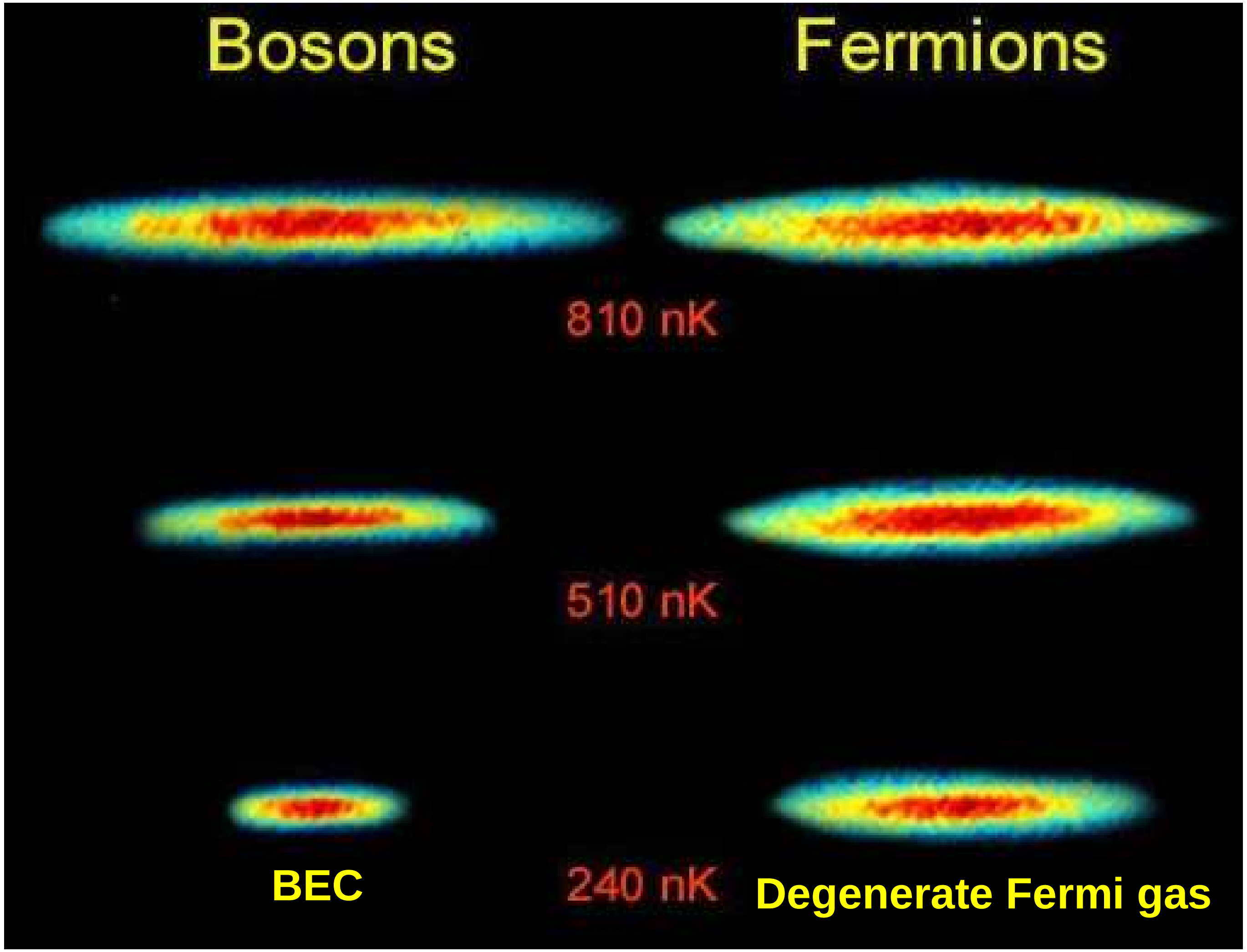
- Condensation is only possible for BOSONS.
- FERMIONS behave differently, due to Pauli.



(Salomon, ENS, 2001)



Observing quantum statistics



(Rice, 2001)

Ultracold fermions

Ideal fermions in a trap

$$V_{ext} = \frac{1}{2} m\left[\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right]$$

if $N >> 1, \quad k_B T >> \hbar\omega_{ho}$ one can use semiclassical approximation

$$f(r,p) = \frac{1}{\exp[(p^2/2m + V_{ext}(r) - \mu)/k_B T] + 1}$$

distribution function

normalization can be written as:

$$N = \iint \frac{dr dp}{(2\pi\hbar)^3} f(r,p) = \frac{1}{2(\hbar\omega_{ho})^3} \int d\varepsilon \, \frac{\varepsilon^2}{\exp[(\varepsilon - \mu)/k_B T] + 1}$$

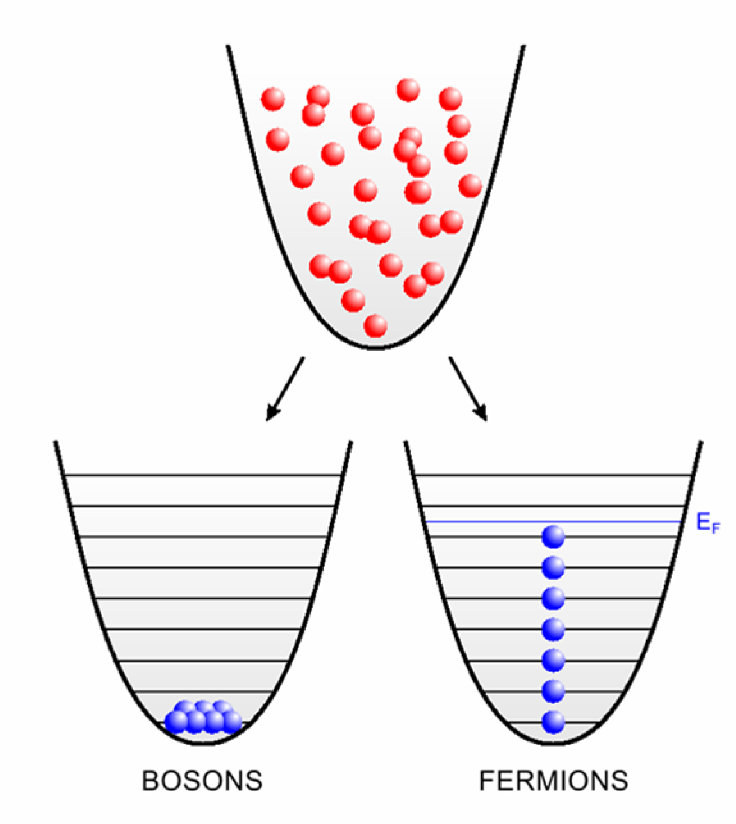
At T=0: $\mu \to E_F; \quad f(r,p) \to \Theta(p^2/2m + V_{ext} - E_F)$ step function

One gets $$E_F = \hbar\omega_{ho}(6N)^{1/3}$$

Same dependence as BEC critical temperature

$$k_B T_c = 0.94\,\hbar\omega_{ho} N^{1/3}$$

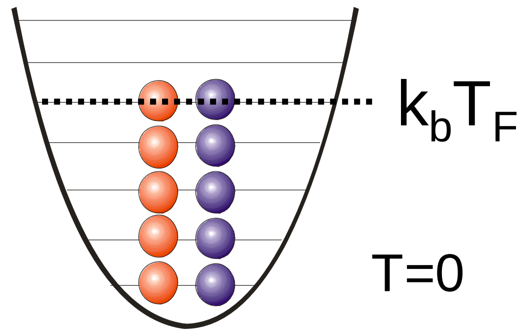
Ultracold fermions



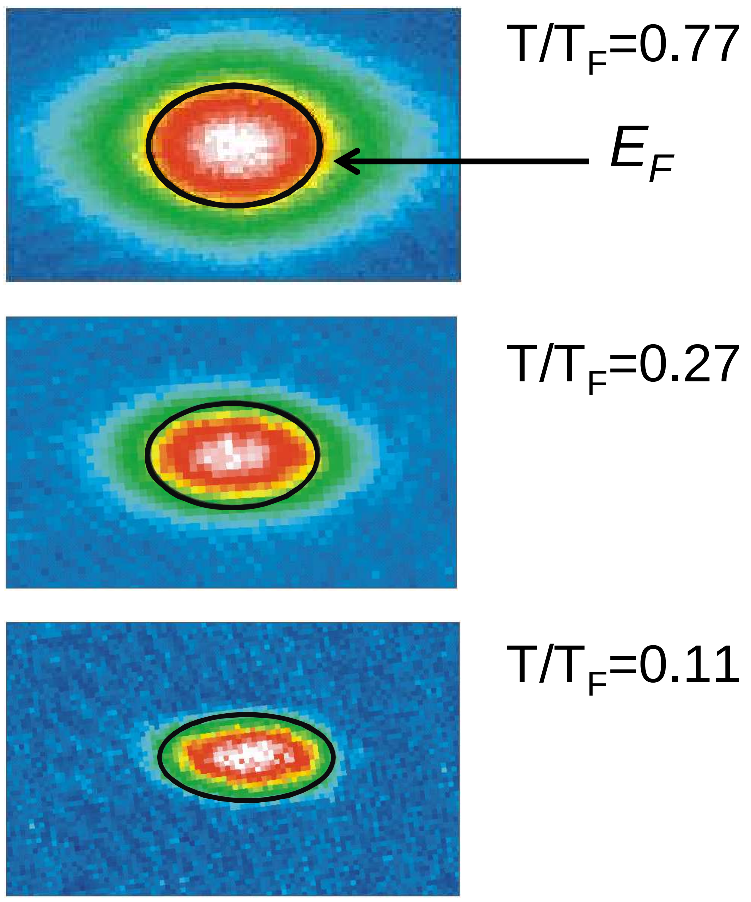
$$k_B T << k_B T_c = 0.94\,\hbar\omega_{ho} N^{1/3}$$

$$k_B T << k_B T_F = E_F = \hbar\omega_{ho} (6N)^{1/3}$$

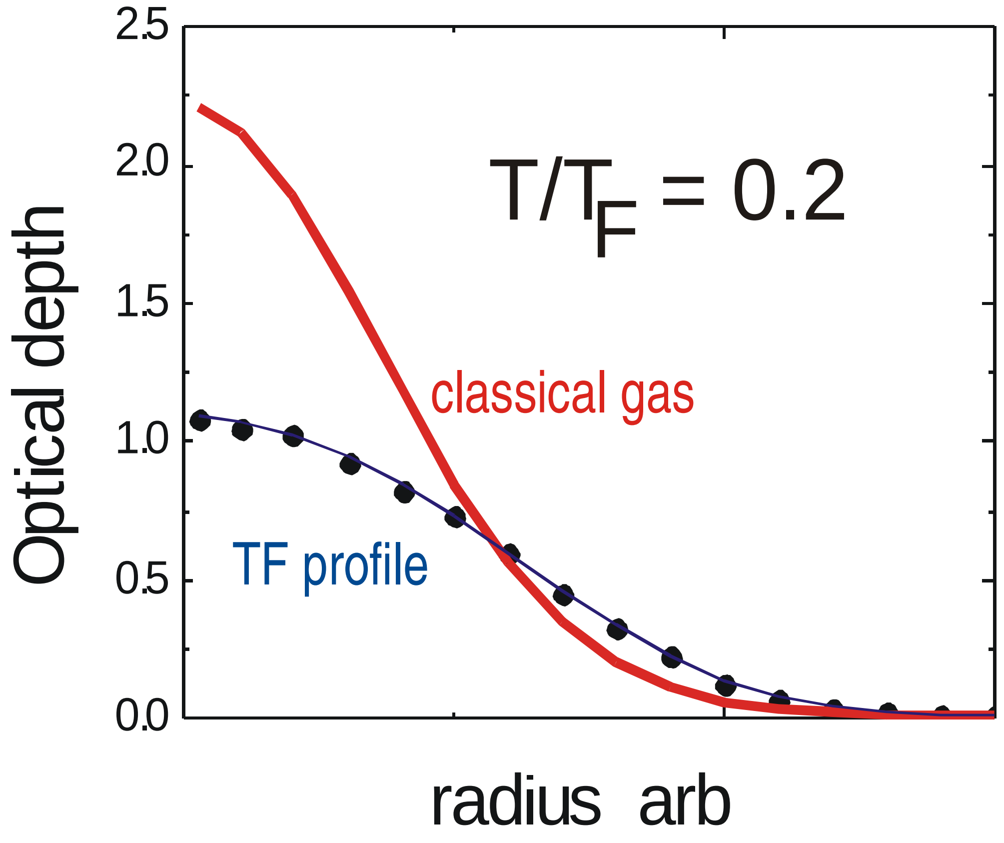
Ultracold fermions



Density profile



(Regal et al., JILA)



Ultracold fermions

Another consequence of Pauli exclusion principle:

Fermions of the **same atomic species** and in the **same spin state** do not interact in s-wave scattering ***!!***

⇩

Just a (almost) free degenerate Fermi gas…

Ultracold fermions

Another consequence of Pauli exclusion principle:

Fermions of the **same atomic species** and in the **same spin state** do not interact in s-wave scattering ***!!***

⇩

Just a (almost) free degenerate Fermi gas…

BUT what about a mixture of two spin states or two species ***?***

- s-wave scattering is possible and dominates at low temperature
- s-wave scattering length can be tuned thanks to Feshbach resonances

Ultracold fermions

Mixture of two spin states or two species

$$\hat{H}_{\text{int}} = g \int d\mathbf{r}\, \hat{\Psi}^{+}_{\uparrow}(\mathbf{r}) \hat{\Psi}^{+}_{\downarrow}(\mathbf{r}) \hat{\Psi}_{\uparrow}(\mathbf{r}) \hat{\Psi}_{\downarrow}(\mathbf{r})$$

with $g = 4\pi\hbar^2 a / m$

a<0 atoms can form bound pairs (bosons) and undergo BCS superfluidity

Ultracold fermions

BCS theory $a < 0$ $k_F |a| << 1$

Pairing between spin-up and -down atoms **in momentum space** **(Cooper pairs).**

Order parameter characterizing the long range order of **two-body** density matrix.

$$\Delta(\mathbf{r}) = \left\langle \hat{\Psi}_{\uparrow}(\mathbf{r}) \hat{\Psi}_{\downarrow}(\mathbf{r}) \right\rangle$$

Quasi-particle excitation spectrum has a **gap:**

$$\varepsilon(p) = \sqrt{\Delta_{gap}^2 + [p^2/2m) - \mu]^2}$$

½ of the energy required to break a pair

BCS **critical temperature**:

$$T_c = 0.28 T_F \exp\left[-\frac{\pi}{2k_F|a|}\right]$$

Note: this prefactor contains Gorkov and Melik-Barkhudarov corrections to BCS (renormalization of scattering length due to screening effects)

Note: this can be very small if $k_F|a|$ is small !

Same physics of weak coupling superconductors!

Ultracold fermions

Mixture of two spin states or two species

$$\hat{H}_{\text{int}} = g \int d\mathbf{r}\, \hat{\Psi}^{+}_{\uparrow}(\mathbf{r}) \hat{\Psi}^{+}_{\downarrow}(\mathbf{r}) \hat{\Psi}_{\uparrow}(\mathbf{r}) \hat{\Psi}_{\downarrow}(\mathbf{r})$$

with $g = 4\pi\hbar^2 a / m$

a<0 atoms can form bound pairs (bosons) and undergo BCS superfluidity

a>0 atoms can form bound molecules (bosons) and undergo BEC.

Ultracold fermions

BEC of molecules $a > 0$ $k_F |a| << 1$

When the scattering length *a* is positive, the interaction can produce weakly **bound molecules** (bosons) of size *a*.

If $k_F|a|$ is small, the size of molecules is much smaller than the average distance between them (gas of bosonic dimers).

By solving the two-body scattering problem one finds the molecular binding energy:

$$E = -\frac{\hbar^2}{ma^2}$$

At low T, molecules form a BEC. The critical temperature for a gas of bosons of mass *2m* (molecules) at density *n* is directly related to value of Fermi energy of fermions of mass *m* at the same density.

In a uniform gas: $T_{BEC} = 0.2 T_F$ In a harmonic trap: $T_{BEC} = 0.5 T_F$

Critical temperature for superfluidity is much **higher in BEC than in BCS** side where it is exponentially small.

Ultracold fermions

Mixture of two spin states or two species

$$\hat{H}_{\text{int}} = g \int d\mathbf{r}\, \hat{\Psi}_{\uparrow}^{+}(\mathbf{r}) \hat{\Psi}_{\downarrow}^{+}(\mathbf{r}) \hat{\Psi}_{\uparrow}(\mathbf{r}) \hat{\Psi}_{\downarrow}(\mathbf{r})$$

with $g = 4\pi\hbar^2 a / m$

a<0 atoms can form bound pairs (bosons) and undergo BCS superfluidity

a>0 atoms can form bound molecules (bosons) and undergo BEC.

In both cases one gets deep modifications of many-body wave function.
Ideal Fermi gas is no longer proper starting point.

⇓

No perturbative theories

Ultracold fermions

Mixture of two spin states or two species

$$\hat{H}_{\text{int}} = g \int d\mathbf{r}\, \hat{\Psi}^+_\uparrow(\mathbf{r}) \hat{\Psi}^+_\downarrow(\mathbf{r}) \hat{\Psi}_\uparrow(\mathbf{r}) \hat{\Psi}_\downarrow(\mathbf{r})$$

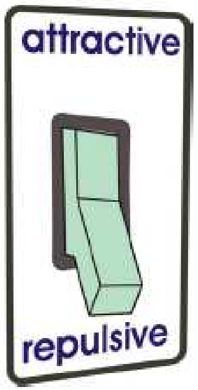
with $g = 4\pi\hbar^2 a / m$

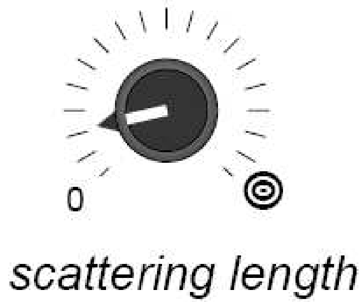
a<0 atoms can form bound pairs (bosons) and undergo BCS superfluidity

a>0 atoms can form bound molecules (bosons) and undergo BEC.

The scattering length can be tuned at will when the atomic species exhibits Feshbach resonances.

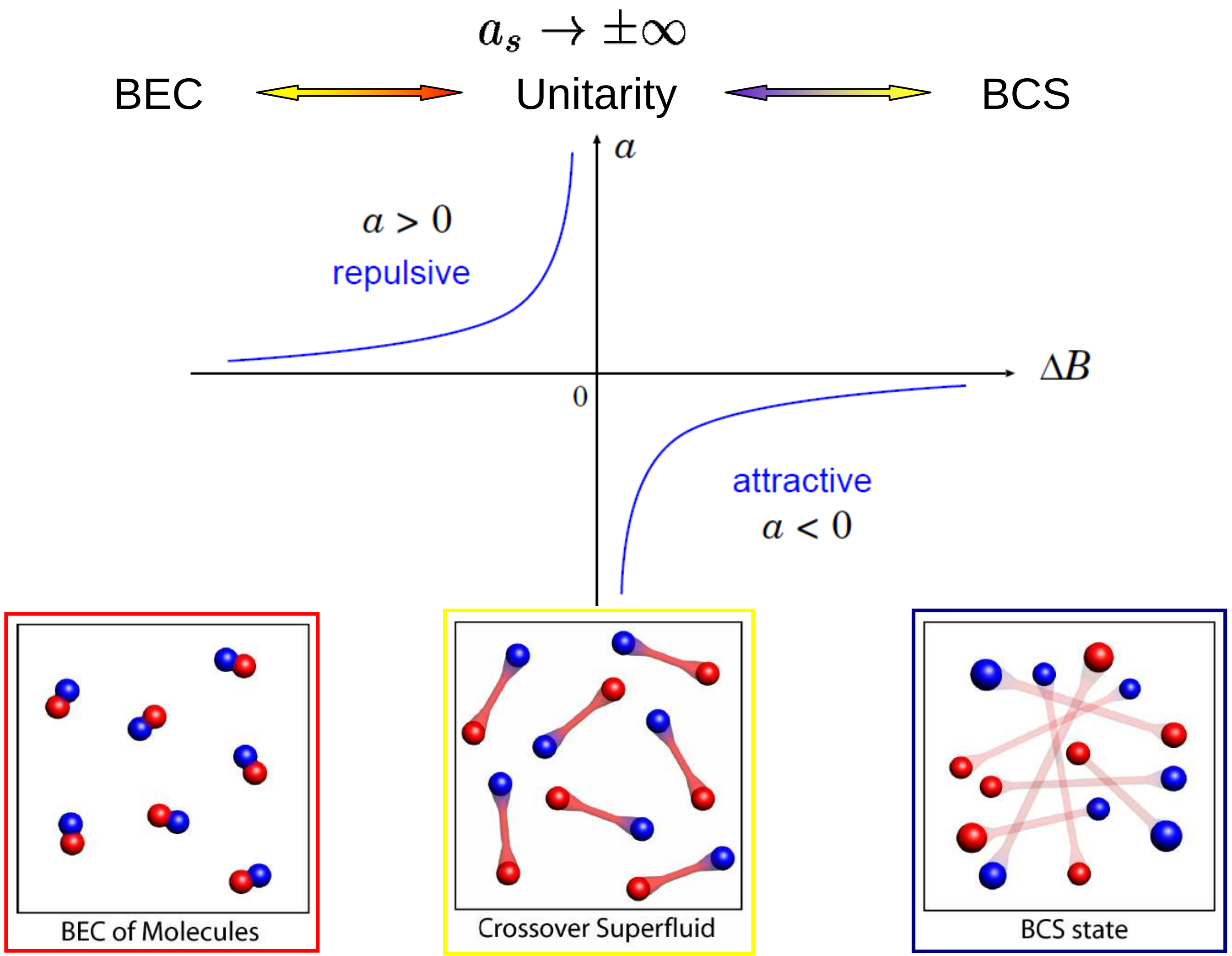
BCS-BEC crossover





BCS-BEC crossover

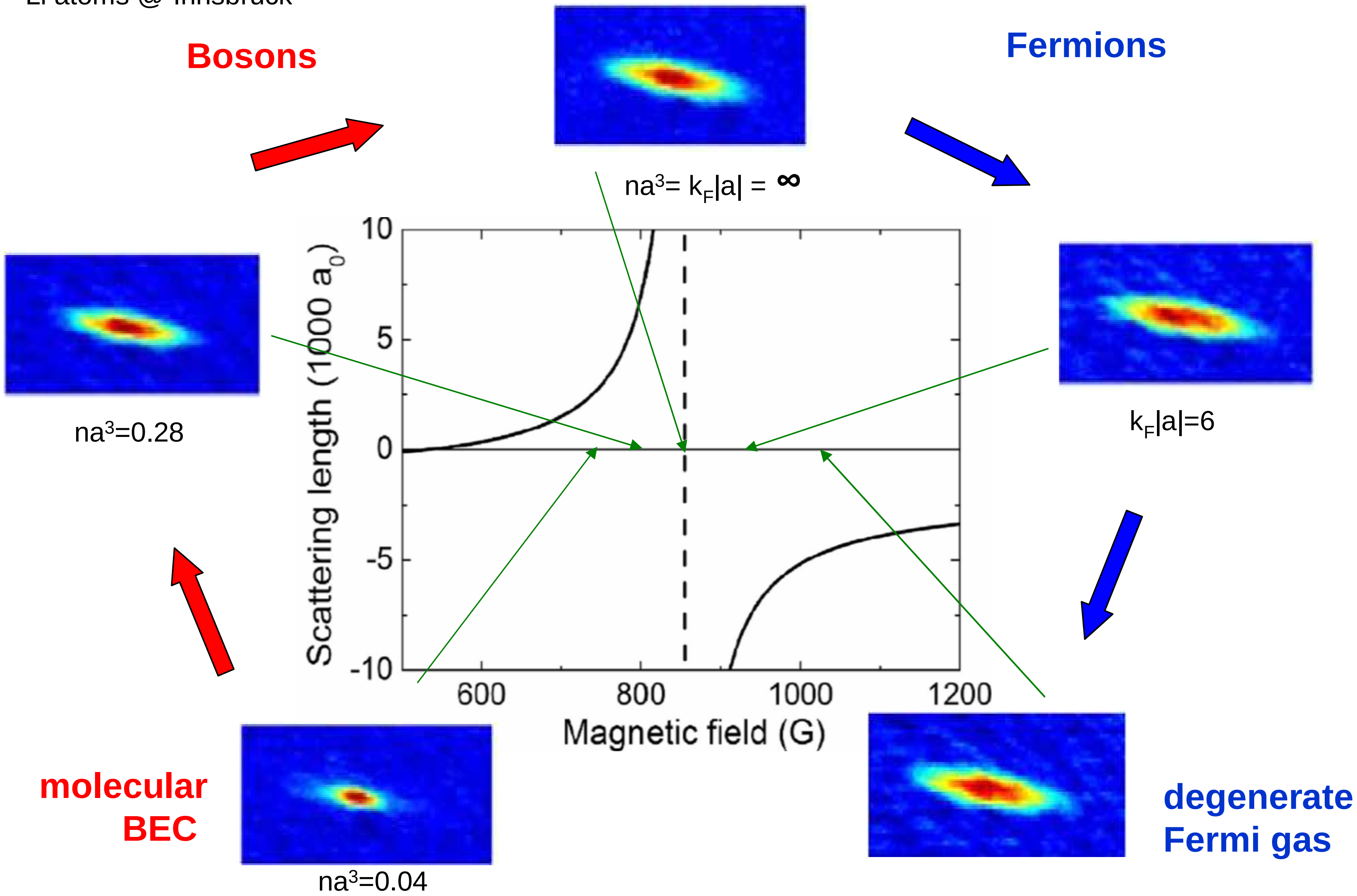
(in a 2-component Fermi gases)



Ultracold fermions

BCS-BEC crossover

^{6}Li atoms @ Innsbruck



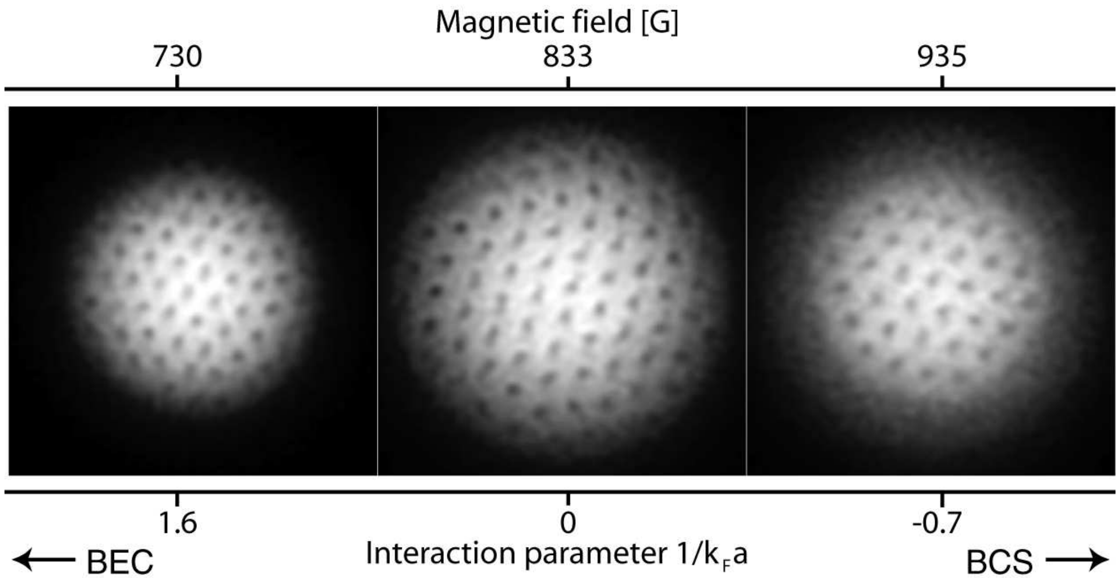
Ultracold fermions

Experiments

^{40}K: JILA (1999)
Florence (2002)
Zurich (2004)
Hamburg (2004)
Toronto (2005)

^{6}Li: Rice (2001)
ENS (2001)
Duke (2001)
MIT (2002)
Innsbruck (2003)

…and many more after 2005.
Also imbalanced spin mixtures (different populations).
Also mixtures of different atoms: 40K-87Rb (Florence), 6Li-23Na (MIT), …



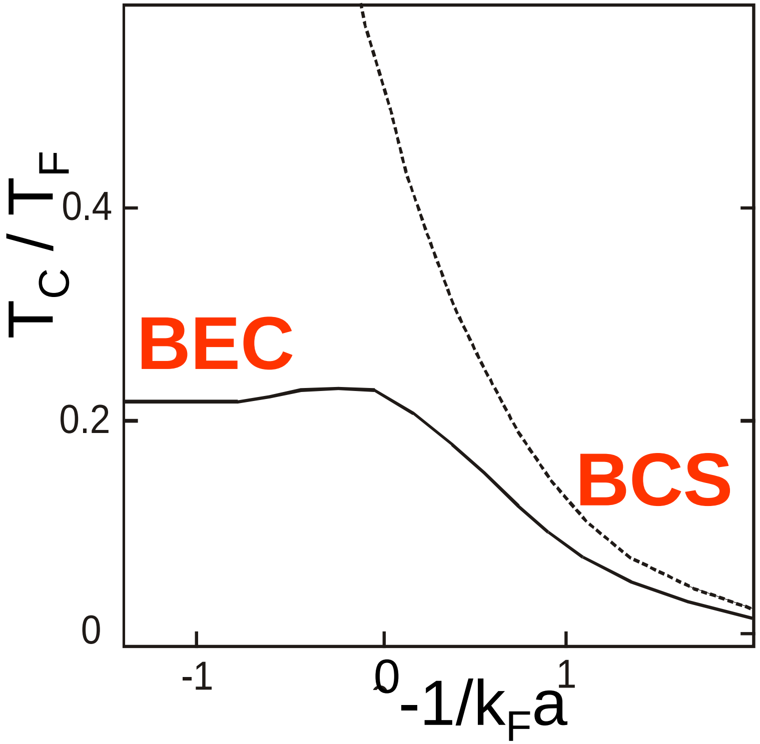
Quantized vortices in the BCS-BEC crossover (MIT 2005)

Theory of the BCS-BEC crossover

First theoretical approach developed by Leggett (1980).
Nozieres and Schmitt-Rink (1985) generalized the gap equation of BCS theory to include the whole resonance region.

Theory predicts (Randeria,1993):

- critical temperature and equation of state as a function of dimensionless the parameter $k_F a$
- formation of molecules with energy $\hbar^2 / ma^2$ on the BEC side $k_F a << 1$
- BEC of molecules interacting with scattering length $a_M = 2a$



Important efforts in recent few years to provide improved **many-body schemes** (Holland, Griffin, Timmermans, Strinati ….), including numerical quantum **Monte Carlo** approaches (Carlson, Giorgini,…)

Theory of the BCS-BEC crossover

Unitary regime when $k_F |a| >> 1$

At **unitarity** the system is strongly correlated but its properties **do not** depend on the value of scattering length **a** (not even on the sign of a)!

⇓

The typical length scale of interaction becomes much larger than the size of the gas itself. It disappears from the description of the system.

⇓

Universality

All lengths disappear from the calculation of energy, chemical potential, thermodynamic functions, etc., except the interparticle distance, which is fixed by the total density of the gas *n*.

Theory of the BCS-BEC crossover

Unitary regime when $k_F |a| >> 1$

Universality

All lengths disappear from the calculation of energy, chemical potential, thermodynamic functions, etc., except the interparticle distance, which is fixed by the total density of the gas *n*.

Example:

The equation of state of a **unitary uniform gas** at T=0 must exhibit the **same** density dependence as the **ideal Fermi gas** (dimensionality arguments rule out different dependences). Thus

$$\mu = (1+\beta)\frac{\hbar^2}{2m}\left(6\pi^2 n\right)^{2/3}$$

with

$\beta = 0$ Ideal Fermi gas

$\beta \neq 0$ Fermi gas at unitarity

Many-body calculations are needed to determine value of β.

One finds: β≈–0.6 (it is negative, reflecting attractive role of the interaction).

The equation of state can be used to determine density profiles, release energy and collective frequencies in Thomas-Fermi approximation.

Mean-field theory of the BCS-BEC crossover

Uniform gas at T=0.
Many-body Hamiltonian written in terms of fermionic operators $\hat{\Psi}^+_\uparrow, \hat{\Psi}^+_\downarrow, \hat{\Psi}_\uparrow, \hat{\Psi}_\downarrow$

$$\hat{H} = \sum_{\sigma=\uparrow,\downarrow} \int d\mathbf{r}\, \hat{\Psi}^+_\sigma(\mathbf{r}) \left[-\frac{\hbar^2\nabla^2}{2m_\sigma} - \mu_\sigma \right] \hat{\Psi}_\sigma(\mathbf{r}) + \frac{1}{2} \iint d\mathbf{r} d\mathbf{r}' \hat{\Psi}^+_\uparrow(\mathbf{r}) \hat{\Psi}^+_\downarrow(\mathbf{r}') V(|\mathbf{r}-\mathbf{r}'|) \hat{\Psi}_\downarrow(\mathbf{r}') \hat{\Psi}_\uparrow(\mathbf{r})$$

Approximation: **include** the interaction only in **pairing** correlations, treated at mean-field level. **Ignore** direct (Hartree) interaction terms proportional to the averages $\left\langle \hat{\Psi}^+_\uparrow \hat{\Psi}_\uparrow \right\rangle \quad \left\langle \hat{\Psi}^+_\downarrow \hat{\Psi}_\downarrow \right\rangle$ ⟶ these would give divergent terms at unitarity

One gets the BCS Hamiltonian

$$\hat{H}_{BCS} = \sum_{\sigma=\uparrow,\downarrow} \int d\mathbf{r}\, \hat{\Psi}^+_\sigma(\mathbf{r}) \left[-\frac{\hbar^2\nabla^2}{2m_\sigma} - \mu_\sigma \right] \hat{\Psi}_\sigma(\mathbf{r}) - \int d\mathbf{r} \left\{ \Delta(\mathbf{r}) \left[\hat{\Psi}^+_\uparrow(\mathbf{r}) \hat{\Psi}^+_\downarrow(\mathbf{r}) - (1/2) \left\langle \hat{\Psi}^+_\uparrow(\mathbf{r}) \hat{\Psi}^+_\downarrow(\mathbf{r}) \right\rangle \right] + H.c. \right\}$$

where $\Delta(\mathbf{r}) = -\int d\mathbf{s}\, V(\mathbf{s}) \left\langle \hat{\Psi}_\downarrow(\mathbf{r}+\mathbf{s}/2) \hat{\Psi}_\uparrow(\mathbf{r}-\mathbf{s}/2) \right\rangle$ order parameter

$\mathbf{s} = |\mathbf{r}-\mathbf{r}'|$

Mean-field theory of the BCS-BEC crossover

$$\hat{H}_{BCS} = \sum_{\sigma=\uparrow,\downarrow} \int d\mathbf{r}\, \hat{\Psi}^+_\sigma(\mathbf{r}) \left[-\frac{\hbar^2\nabla^2}{2m_\sigma} - \mu_\sigma \right] \hat{\Psi}_\sigma(\mathbf{r}) - \int d\mathbf{r} \left\{ \Delta(\mathbf{r}) \left[\hat{\Psi}^+_\uparrow(\mathbf{r}) \hat{\Psi}^+_\downarrow(\mathbf{r}) - (1/2) \left\langle \hat{\Psi}^+_\uparrow(\mathbf{r}) \hat{\Psi}^+_\downarrow(\mathbf{r}) \right\rangle \right] + H.c. \right\}$$

$$\Delta(\mathbf{r}) = -\int d\mathbf{s}\, V(\mathbf{s}) \left\langle \hat{\Psi}_\downarrow(\mathbf{r}+\mathbf{s}/2) \hat{\Psi}_\uparrow(\mathbf{r}-\mathbf{s}/2) \right\rangle$$

Crucial point: this Hamiltonian can be put in diagonal form by replacing particles with **quasi-particles** (**Bogoliubov transformations**) **!**

$$\hat{\Psi}_\uparrow(\mathbf{r}) = \sum_i [u_i(\mathbf{r})\hat{\alpha}_i + v_i^*(\mathbf{r})\hat{\beta}_i^+]$$

$$\hat{\Psi}_\downarrow(\mathbf{r}) = \sum_i [u_i(\mathbf{r})\hat{\beta}_i + v_i^*(\mathbf{r})\hat{\alpha}_i^+]$$

$$\hat{H}_{BCS} = (E_0 - \mu N) + \sum_i \varepsilon_i (\hat{\alpha}_i^+ \hat{\alpha}_i + \hat{\beta}_i^+ \hat{\beta}_i)$$

a gas of independent quasi-particles

The quasi-particle operators obey the anti-commutation rules $\left\{\hat{\alpha}_i, \hat{\alpha}_j^+\right\} = \left\{\hat{\beta}_i, \hat{\beta}_j^+\right\} = \delta_{ij}$

The quasiparticle amplitudes obey $\int d\mathbf{r}\, [u_i^*(\mathbf{r}) u_j(\mathbf{r}) + v_i^*(\mathbf{r}) v_j(\mathbf{r})] = \delta_{ij}$

Mean-field theory of the BCS-BEC crossover

$$\hat{\Psi}_{\uparrow}(\mathbf{r}) = \sum_i [u_i(\mathbf{r})\hat{\alpha}_i + v_i^*(\mathbf{r})\hat{\beta}_i^+]$$
$$\hat{\Psi}_{\downarrow}(\mathbf{r}) = \sum_i [u_i(\mathbf{r})\hat{\beta}_i + v_i^*(\mathbf{r})\hat{\alpha}_i^+]$$

The diagonalization of the Hamiltonian gives the equations for the quasi-particle amplitudes. In general, including the case of a Fermi gas in an external potential, these equations have the form

$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

Bogoliubov – de Gennes equations

with $H_0(\mathbf{r}) = -\dfrac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu$

The order parameter can be obtained by means of a self-consistent procedure. A proper regularization of the interaction is needed (see Giorgini et al. RMP 2008 for details).

Mean-field theory of the BCS-BEC crossover

Bogoliubov – de Gennes equations

Fermions

$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$H_0(\mathbf{r}) = -\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu$$

Note: remarkable similarity with **Bogoliubov** equations for excitations (quasi-particles) in BECs

Bosons

$$\begin{pmatrix} H_0(\mathbf{r}) & g\Psi_0^2 \\ -g\Psi_0^{*2}(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$H_0(\mathbf{r}) = -\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu + 2g\left|\Psi_0(\mathbf{r})\right|^2$$

Note: in lecture #2 we wrote the same eqs in this form:

$$\hbar\omega_j u_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)u_j + g\Psi_0^2 v_j$$

$$-\hbar\omega_j v_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)v_j + g\Psi_0^{*2} u_j$$

Mean-field theory of the BCS-BEC crossover

Fermions vs. **Bosons**

Bogoliubov equations for BECs and Bogoliubov-de Gennes for fermions are two implementations of the same idea: Bogoliubov transformation, i.e. , diagonalization of the many-body Hamiltonian by replacing particle operators by quasi-particle operators.

→ one of the key concepts in many-body theories

Mean-field theory of the BCS-BEC crossover

Fermions vs. **Bosons**

Bogoliubov equations for BECs and Bogoliubov-de Gennes for fermions are two implementations of the same idea: Bogoliubov transformation, i.e. , diagonalization of the many-body Hamiltonian by replacing particle operators by quasi-particle operators.

Important difference:

Due to macroscopic occupation of a single state, in BEC the order parameter is obtained expanding the Hamiltonian in $\delta\hat{\Psi}(\mathbf{r})$ where $\hat{\Psi}(\mathbf{r}) = \Psi_0(\mathbf{r}) + \delta\hat{\Psi}(\mathbf{r})$.
At zero-order one has the GP equation for $\Psi_0(\mathbf{r})$.
At first-order one gets the Bogoliubov equations for bosonic excitations.

Conversely, in the BCS-BEC theory for fermions, there is no zero-order. The Bogoliubov-de Gennes equations give the order parameter itself, together with all fermionic excitations (but no bosonic excitations, such as phonons…). The ground state is the vacuum of quasi-particles.

Mean-field theory of the BCS-BEC crossover

Bogoliubov – de Gennes equations

$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

Fermions

$$H_0(\mathbf{r}) = -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu$$

The mean-field theory based on BdG equation

- gives the correct limit of free fermions in external potentials for $a=0^-$
- gives the correct GP equation for a BEC of molecules of mass *2m* for $a=0^+$
- gives a smooth crossover from BCS to BEC, including unitarity.
- it is accurate enough for many purposes.
- It is misses important corrections to the ideal Fermi gas for small and negative *a*.
- It gives the wrong scattering length for molecule-molecule interaction.

An application: fermions in a 1D optical lattice

Bogoliubov – de Gennes equations

Fermions

$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$H_0(\mathbf{r}) = -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu$$

$$\int d^3r \left[u_i^*(\mathbf{r}) u_j(\mathbf{r}) + v_i^*(\mathbf{r}) v_j(\mathbf{r})\right] = \delta_{i,j}$$

$$n = (2/V) \sum_i \int |v_i(\mathbf{r})|^2 \, d\mathbf{r}$$ ⇐ density

$$\Delta(\mathbf{r}) = -g \sum_i u_i(\mathbf{r}) v_i(\mathbf{r})^*$$ ⇐ order parameter

These equations must be solved numerically by means of an iterative procedure in order to ensure self-consistency.

An application: fermions in a 1D optical lattice

Bogoliubov – de Gennes equations

Fermions

$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$H_0(\mathbf{r}) = -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu$$

$$\int d^3r \, [u_i^*(\mathbf{r}) u_j(\mathbf{r}) + v_i^*(\mathbf{r}) v_j(\mathbf{r})] = \delta_{i,j}$$

$$n = (2/V) \sum_i \int |v_i(\mathbf{r})|^2 \, d\mathbf{r}$$ ⇐ density

$$\Delta(\mathbf{r}) = -g \sum_i u_i(\mathbf{r}) v_i(\mathbf{r})^*$$ ⇐ order parameter

Caveat: a regularization procedure must be used to cure ultraviolet divergence (pseudo-potential, cut-off energy) as discussed by Randeria and Leggett.
See also: G. Bruun et al., Eur. Phys. J D 7, 433 (1999), A. Bulgac and Y. Yu, PRL 88, 042504 (2002).

An application: fermions in a 1D optical lattice

Bogoliubov – de Gennes equations

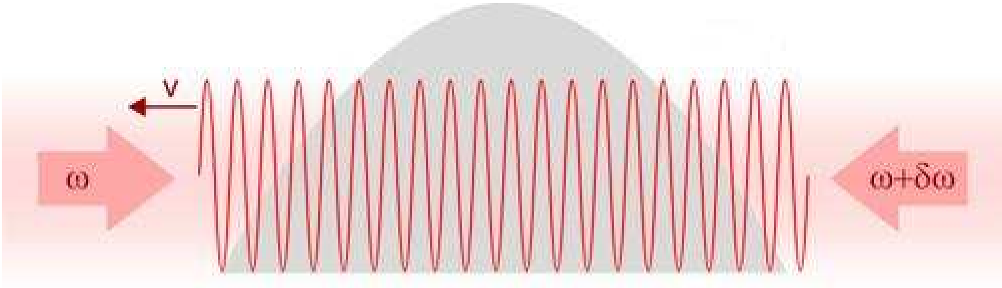
$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

Fermions

$$H_0(\mathbf{r}) = -\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu$$

In a lattice, the external potential is periodic of period *d* and one can use the Bloch wave decomposition

$$u_i(\mathbf{r}) = \tilde{u}_i(z)e^{iPz}e^{i\mathbf{k}\cdot\mathbf{r}}$$
$$v_i(\mathbf{r}) = \tilde{v}_i(z)e^{-iPz}e^{i\mathbf{k}\cdot\mathbf{r}}$$



$$Q = P/\hbar$$

$$\Delta(\mathbf{r}) = e^{i2Qz}\tilde{\Delta}(z)$$

functions of period *d*

Once BdG equations are solved, one can calculate the energy density $e = E/V$

$$e = \int d\mathbf{r} \left[\sum_i 2(\mu - \epsilon_i)|\tilde{v}_i(z)|^2 + \sum_i \Delta(\mathbf{r})\tilde{u}_i(z)\tilde{v}_i^*(z) \right]$$

An application: fermions in a 1D optical lattice

From the energy density one can then calculate

$$\mu = \frac{\partial e(n,P)}{\partial n}$$ chemical potential

$$\kappa^{-1} = n\frac{\partial \mu(n,P)}{\partial n}$$ inverse compressibility

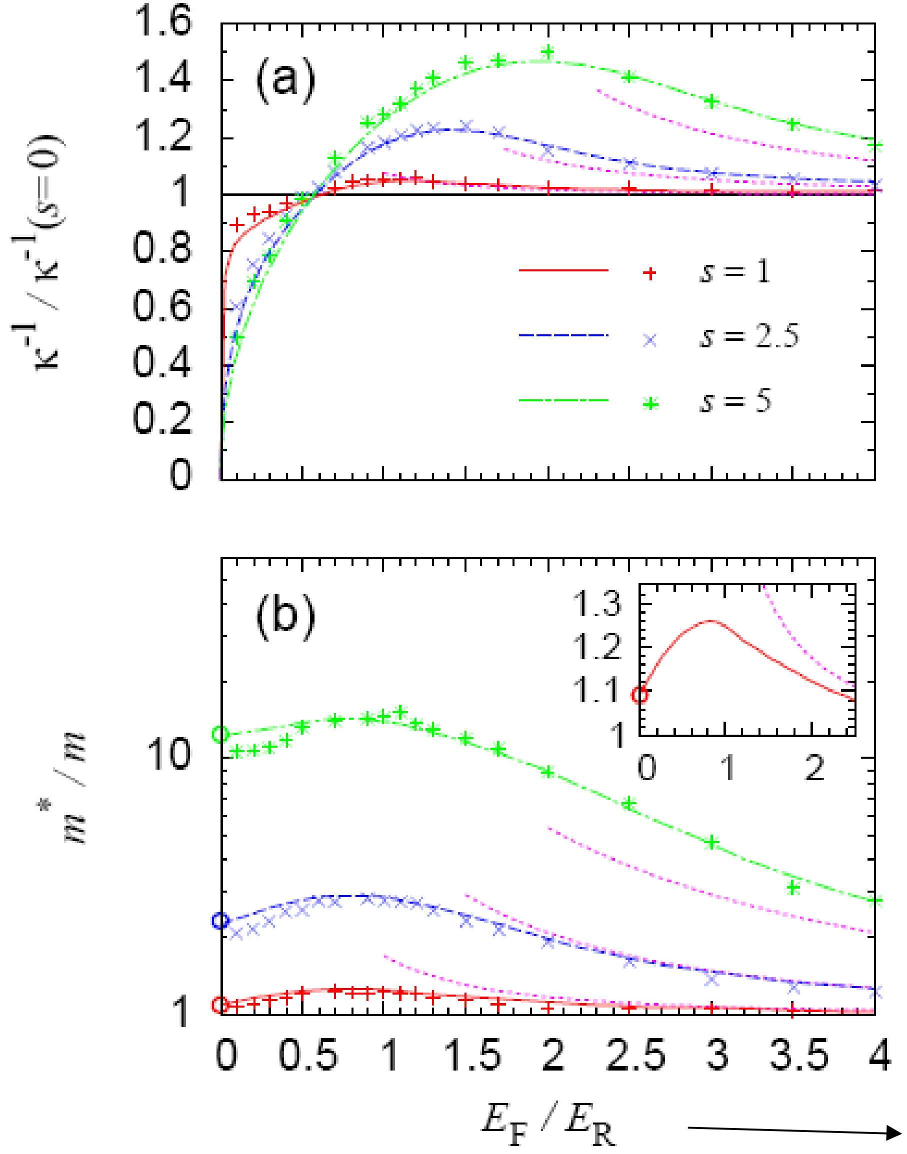
$$\frac{1}{m^*} = \frac{1}{n}\frac{\partial^2 e(n,P)}{\partial P^2}$$ effective mass

$$c_s = (\kappa^{-1}/m^*)^{1/2}$$ sound velocity

Key quantities for the characterization of the collective properties of the superfluid

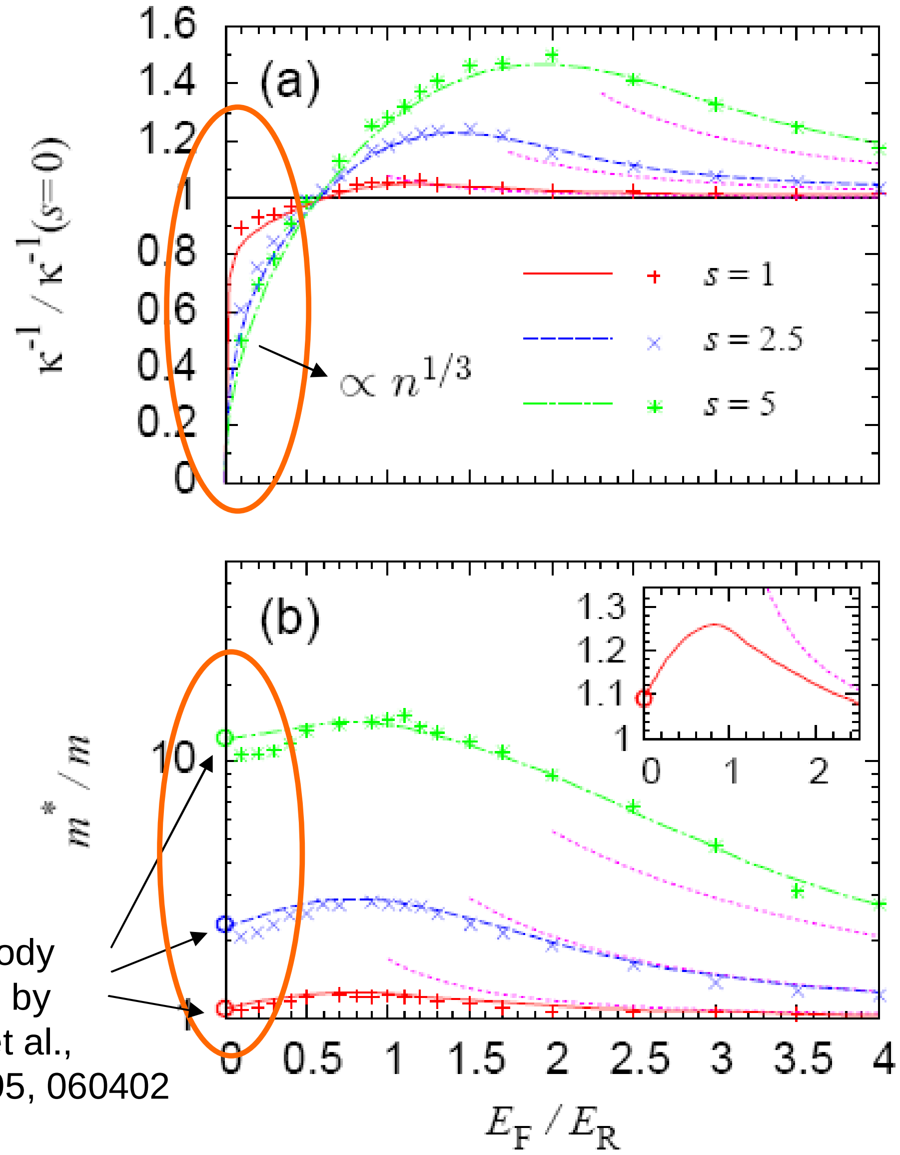
I will show some of the results obtained in:
Equation of state and effective mass of the unitary Fermi gas in a 1D periodic potential [Phys. Rev. A 78, 063619 (2008)].

Results: compressibility and effective mass



$k_F = (3\pi^2 n)^{1/3}$ and $E_F = \hbar^2 k_F^2/(2m)$

Results: compressibility and effective mass



Two-body results by Orso et al., PRL 95, 060402 (2005)

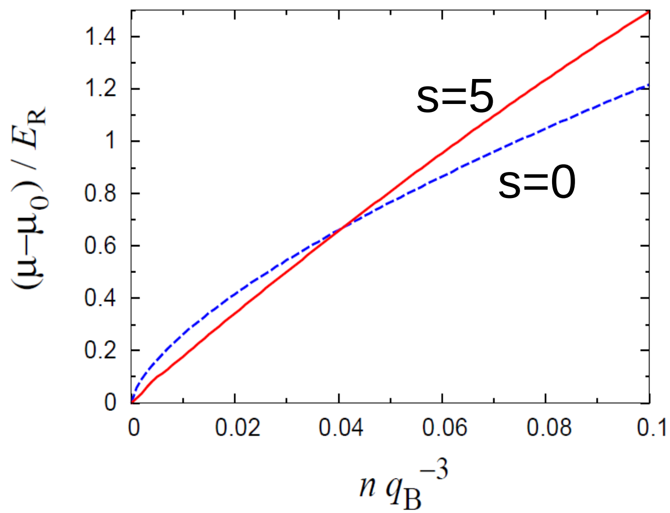
when $E_F << E_R$:

The lattice favors the formation of molecules (bosons).

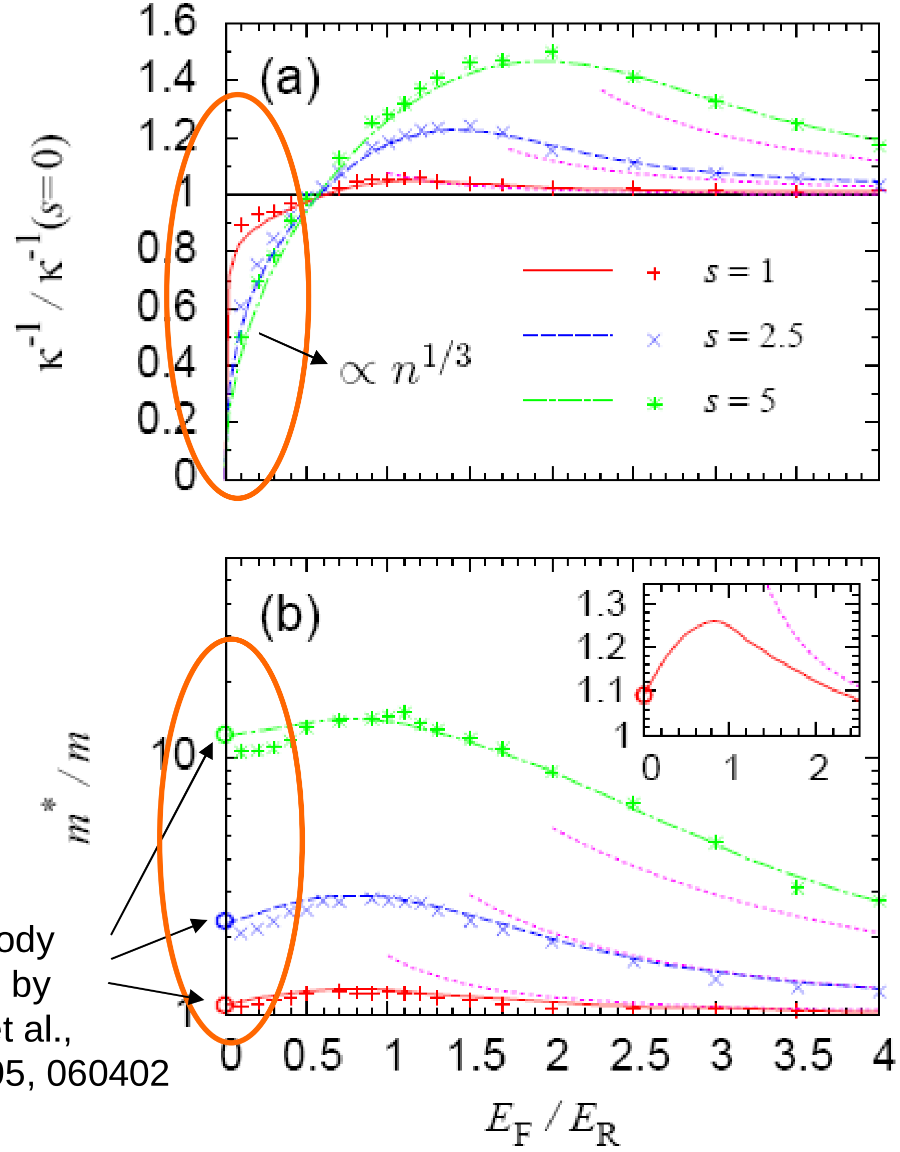
The interparticle distance becomes larger than the molecular size.

In this limit, the BdG equations describe a BEC of molecules.

The chemical potential becomes linear in density.



Results: compressibility and effective mass



Two-body results by Orso et al., PRL 95, 060402 (2005)

when $E_F \ll E_R$:

The lattice favors the formation of molecules (bosons).

The interparticle distance becomes larger than the molecular size.

In this limit, the BdG equations describe a BEC of molecules.

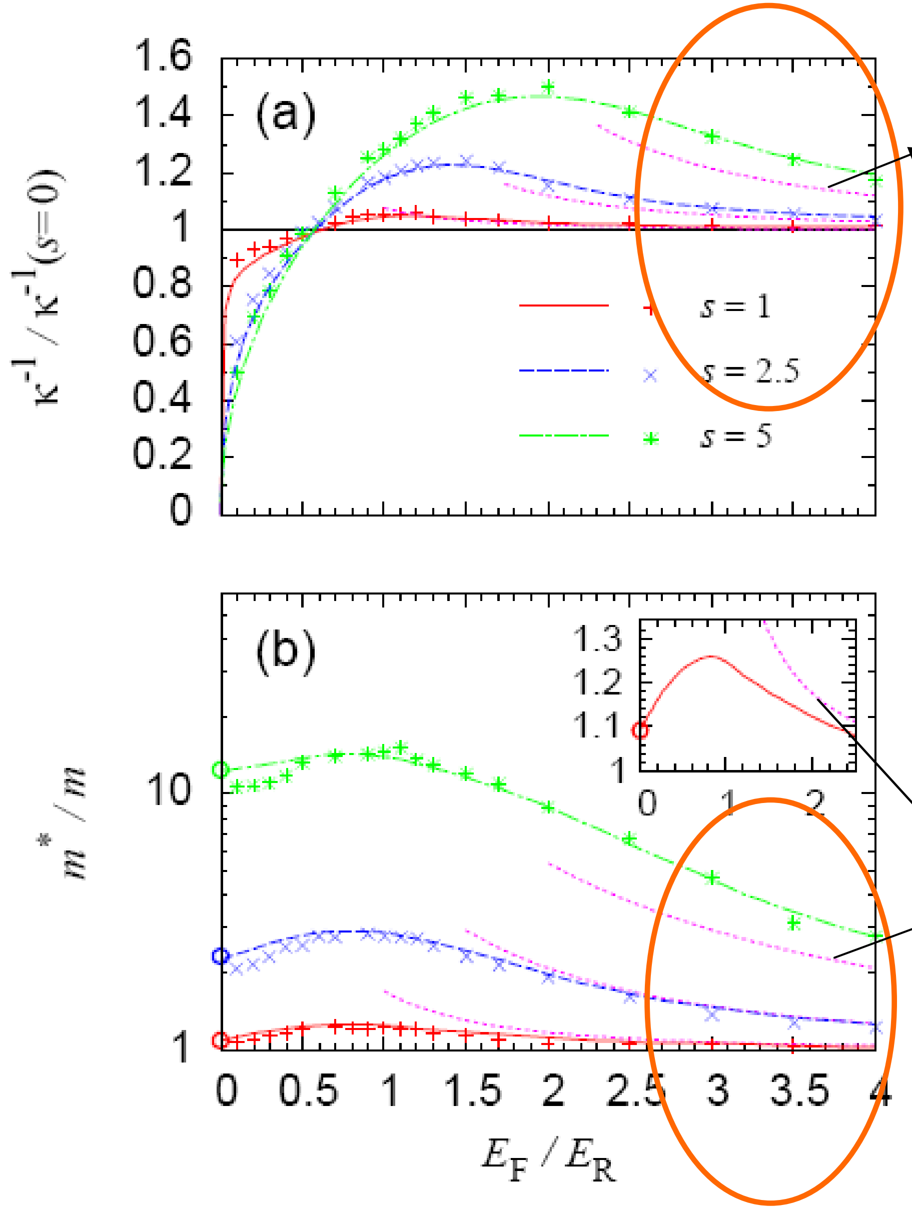
The chemical potential becomes linear in density.

The system is highly compressible.

The effective mass approaches the solution of the two-body problem.

The effects of the lattice are larger than for bosons!

Results: compressibility and effective mass



$$\kappa^{-1} \simeq \frac{2}{3}(1+\beta)E_\mathrm{F}\left[1+\frac{1}{32}\frac{s^2}{(1+\beta)^2(E_\mathrm{F}/E_\mathrm{R})^2}\right]$$

when $E_F >> E_R$:

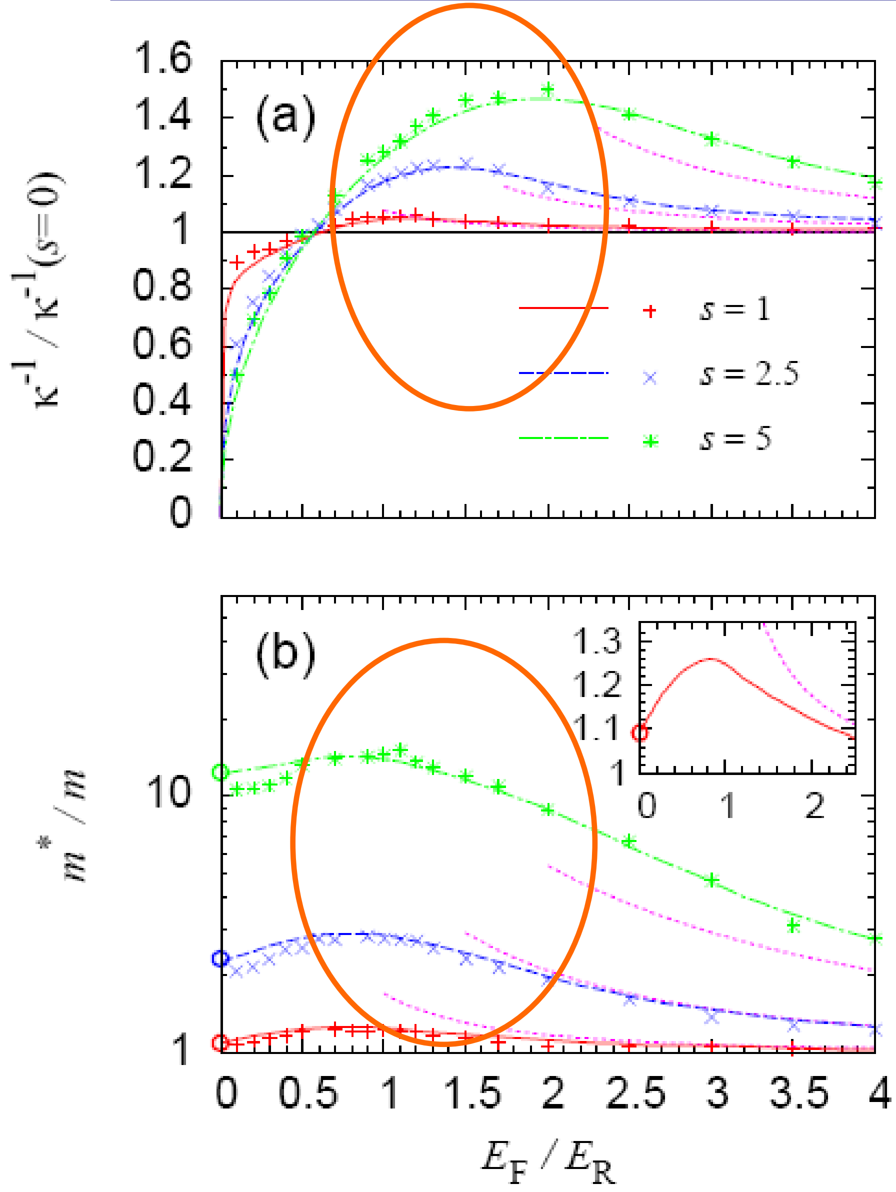
Both quantities approach their values for a uniform gas.

Analytic expansions in the small parameter (sE_R/E_F)

$$\frac{m^*}{m} \simeq 1+\frac{9}{32}\frac{s^2}{(1+\beta)^2(E_\mathrm{F}/E_\mathrm{R})^2}$$

$$e(n,0) = (1+\beta)e^0(n,0)$$

Results: compressibility and effective mass

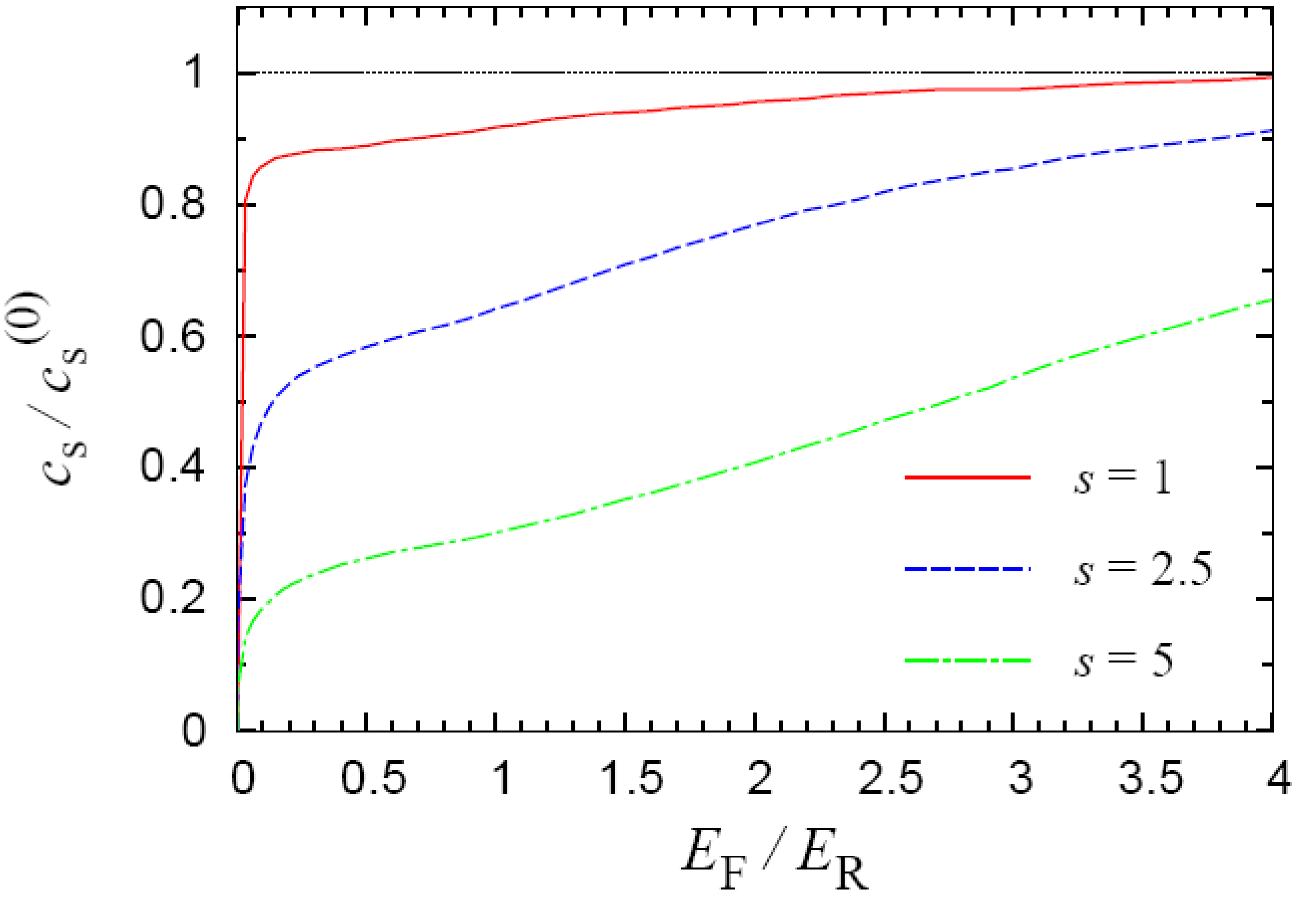


when $E_F \sim E_R$:

Both quantities have a maximum, caused by the band structure of the quasiparticle spectrum.

Sound velocity

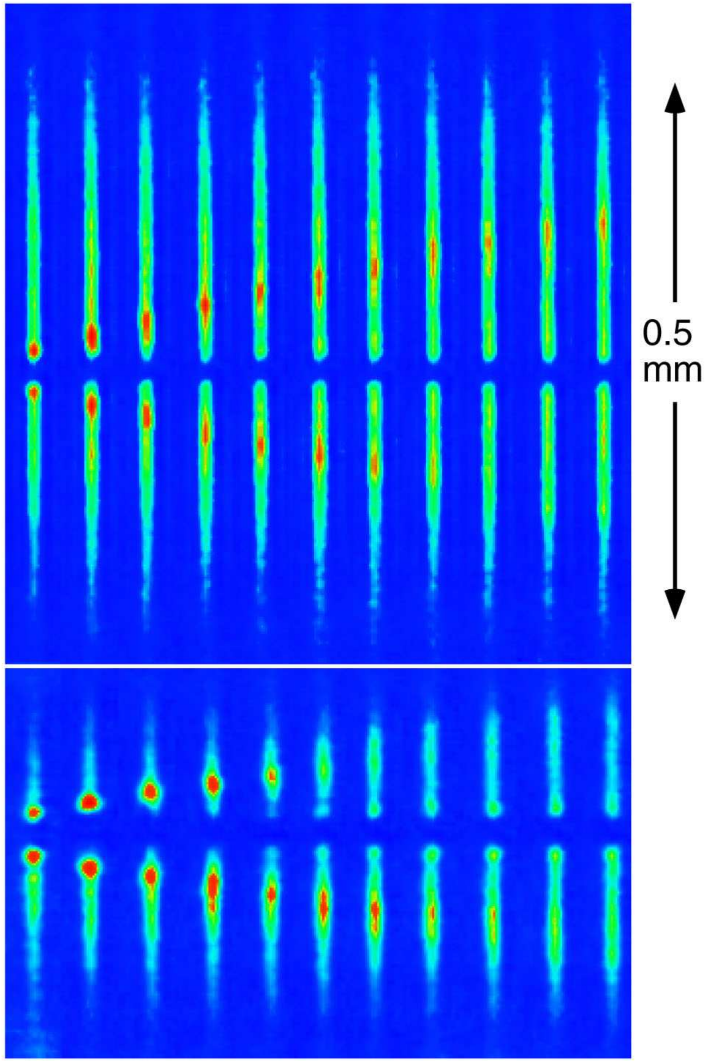
$$c_s = \sqrt{\kappa^{-1}/m^*}$$

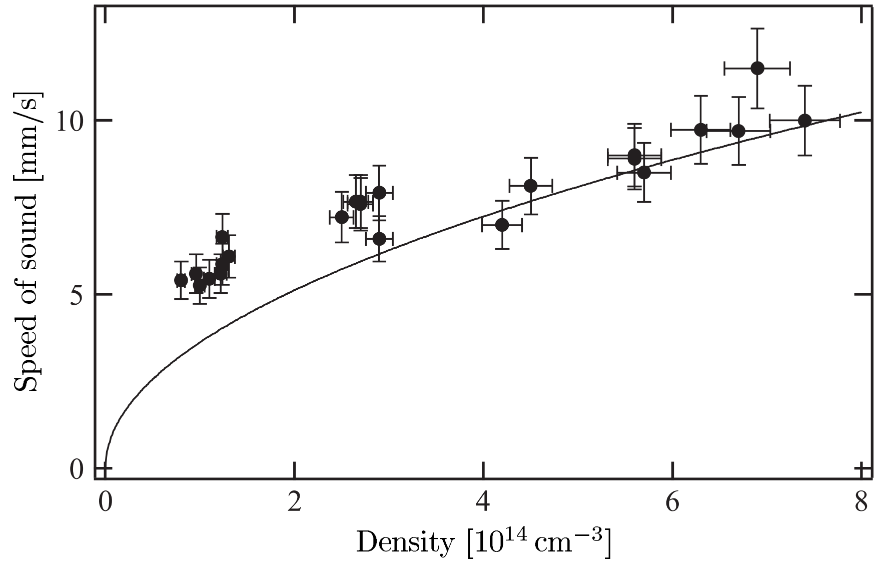


Significant reduction of sound velocity the by lattice !

Sound velocity

$$c_s = \sqrt{\kappa^{-1}/m^*}$$





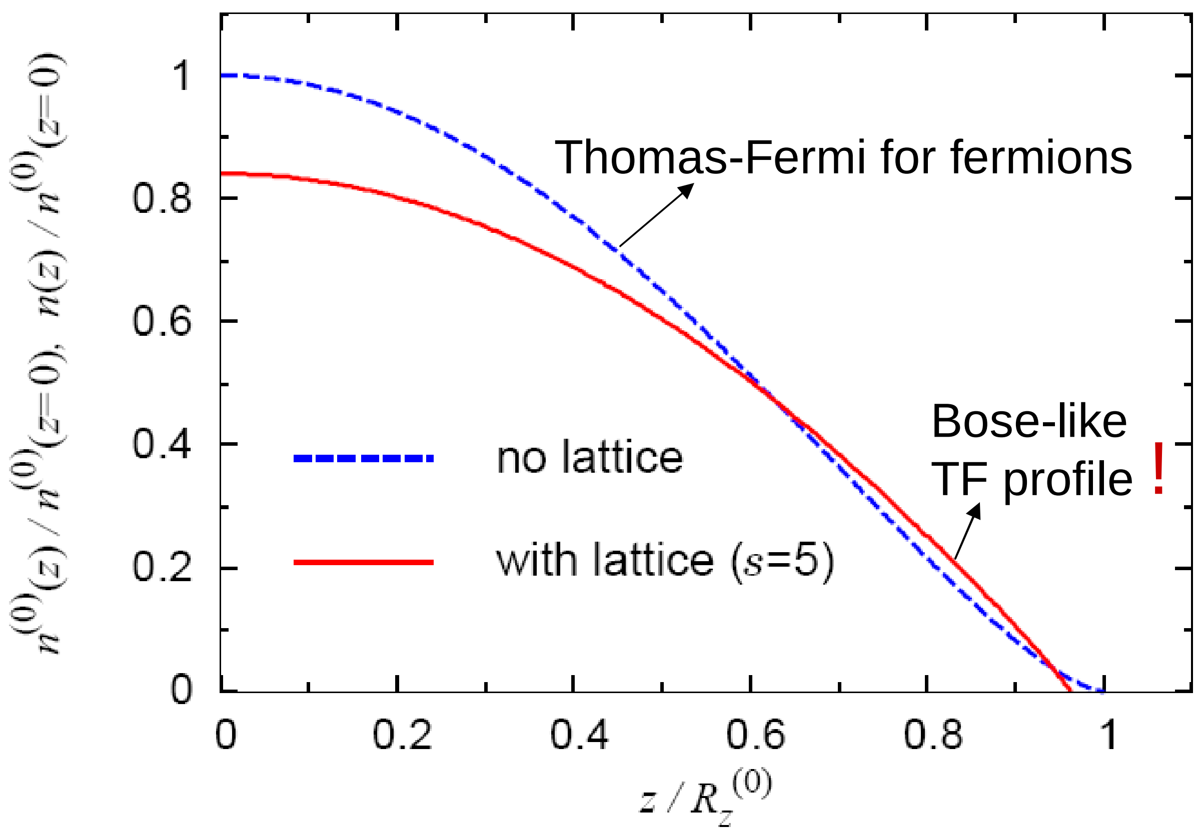
This was the experiment with bosons (MIT, 1997).

One might do the same with fermions with and without optical lattice.

Density profile of a trapped gas

From the results for *μ(n)* and using a local density approximation, we find the density profile of the gas in the harmonic trap + 1D lattice

aspect ratio = 1
$\hbar\omega/E_R = 0.01$
$N=5\times10^5$
s=5



Mean-field theory of the BCS-BEC crossover

Bogoliubov – de Gennes equations

Fermions

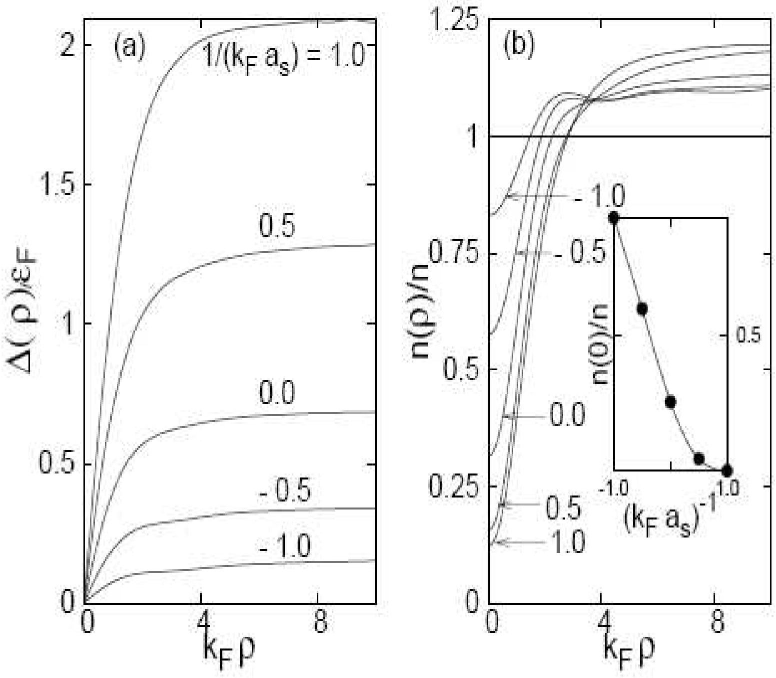
$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$H_0(\mathbf{r}) = -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu$$

We have just seen an example of BdG calculations:
Fermions at unitarity in an optical lattice.

Similar calculations:

Quantized vortex in the BCS-BEC crossover
(Sensarma, Randeria,Ho, PRL 2006).



Mean-field theory of the BCS-BEC crossover

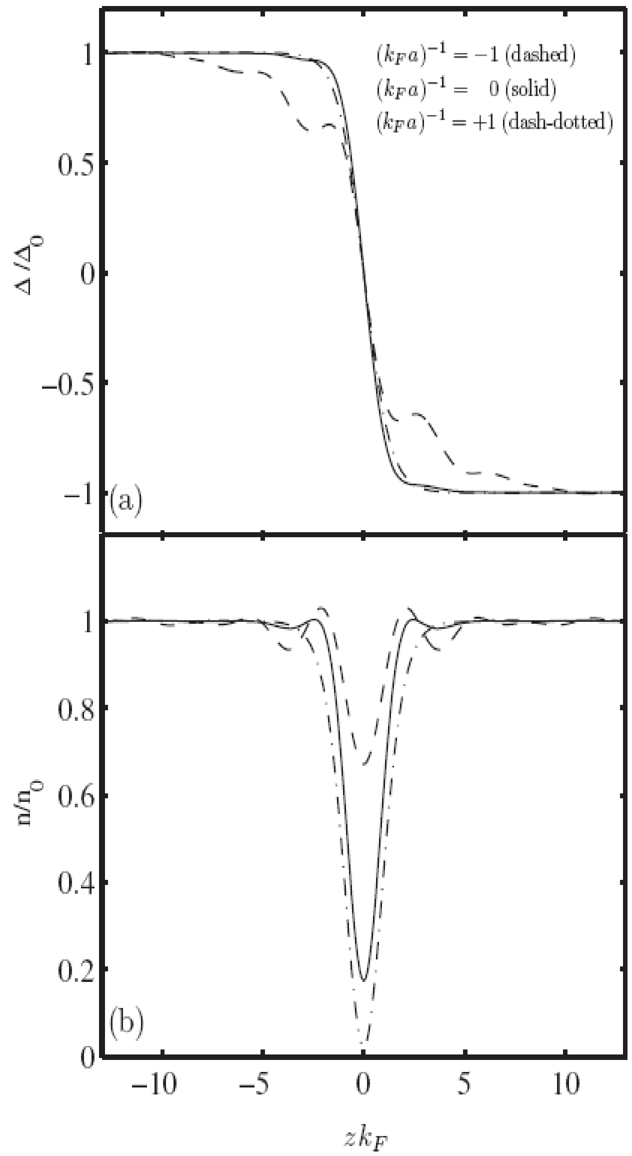
Bogoliubov – de Gennes equations

$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

We have just seen an example of BdG calculations: Fermions at unitarity in an optical lattice.

Similar calculations:

Dark soliton in the BCS-BEC crossover (Trento, PRA 2007).



Mean-field theory of the BCS-BEC crossover

Bogoliubov – de Gennes equations

$$\begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \varepsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

We have just seen an example of BdG calculations:
Fermions at unitarity in an optical lattice.

More recently:

Critical velocity of superfluid flow through single barrier and periodic potentials
(Trento, just submitted).

Mean-field theory of trapped atomic cold gases

Thank you for your attention