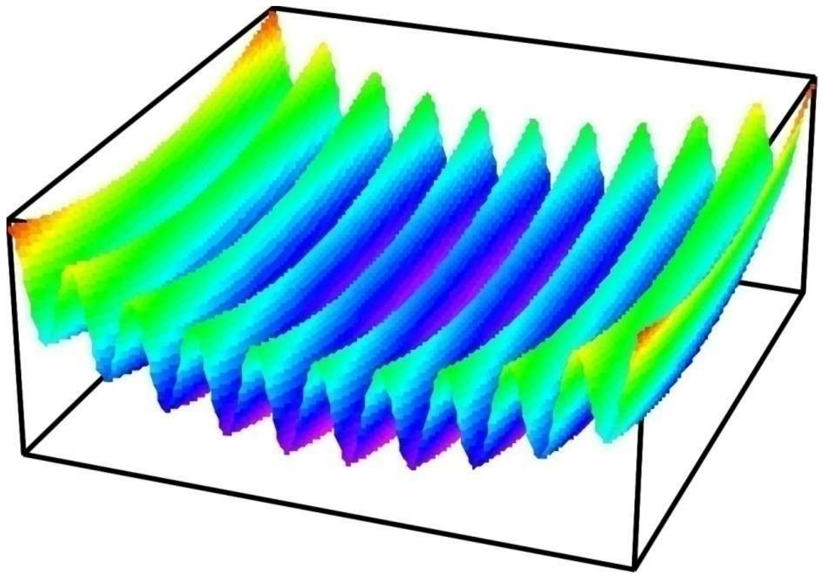
BECs in optical lattices

1D lattice + harmonic trap



A periodic potential can be generated by two counter-propagating laser beams which produce a standing wave of the form $E(z,t) = E e^{-i\omega t}\sin(qz) + c.c.$

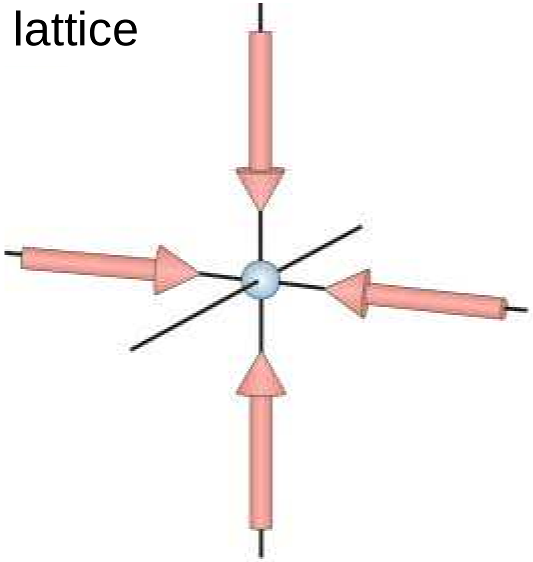
The time averaged effective field $V_{opt}(z) = -(1/2)\alpha(\omega)\langle E^2(z,t)\rangle$ takes the form

$$V_{opt}(z) = -\alpha(\omega)E^2\sin^2(qz)$$

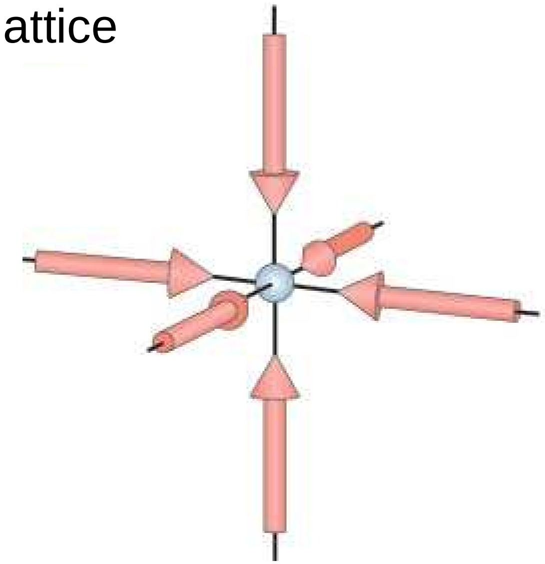
where $\alpha(\omega) \equiv$ dipole polarizability.

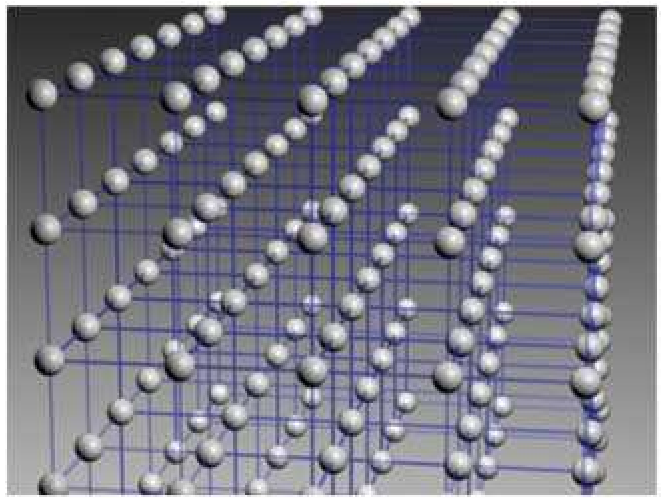
(natural extension to 2D and 3D periodic potentials)

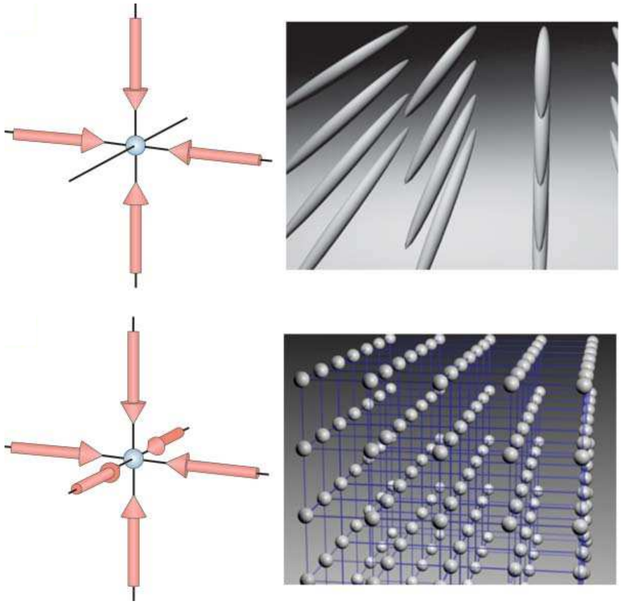
2D lattice



3D lattice







Ideal crystal-like systems:

- no impurities
- no defects
- bosons, fermions, or both together.
- possibility of tuning depth of the potential, lattice spacing, atom-atom interaction, dimensionality.

New physics in the presence of periodic potentials.

-**Without interaction:**
Interference in momentum distribution, Bloch oscillations, etc.

- **With interactions**:
Josephson oscillations, dynamic instabilities, superfluid-Mott insulator transition and other quantum phases (including spin degrees of freedom).

Sort of “Solid state physics” revisited !

BEC in 1D optical lattice

Important length scale: **recoil energy**

Bragg wavevector

$$q_B = \pi / d$$

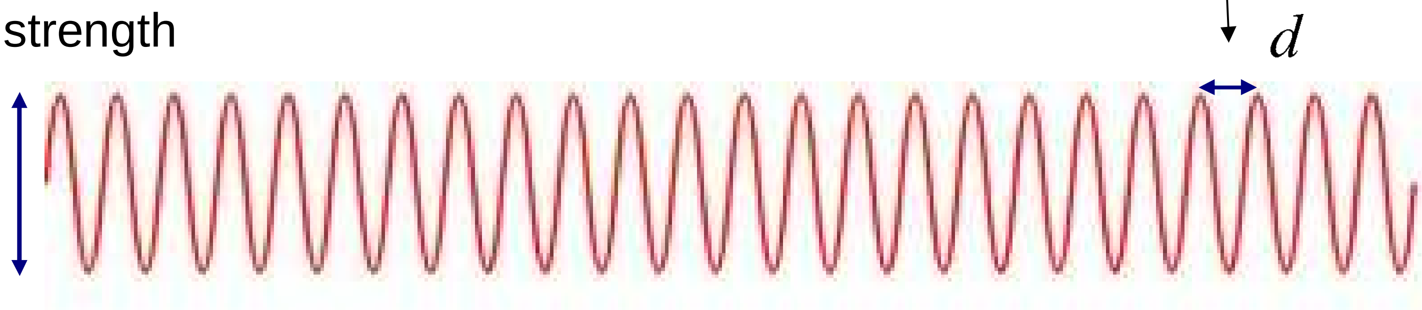
$$E_r = \frac{\hbar^2 q_B^2}{2m} = \frac{\hbar^2 \pi^2}{2md^2}$$

The external potential can be written as

$$V_{opt}(z) = sE_r \sin^2(qz)$$

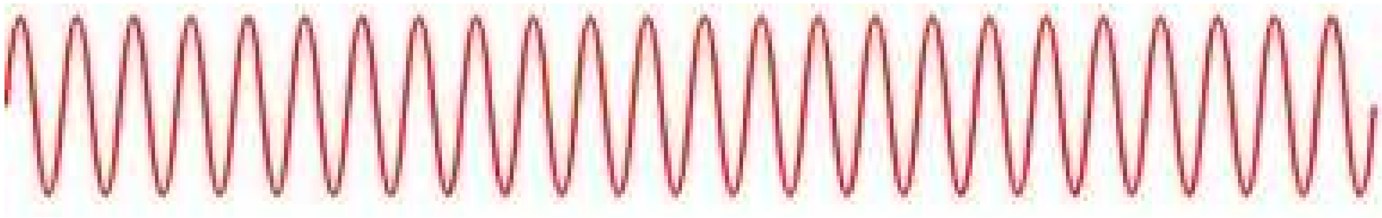
lattice spacing

lattice strength



BEC in 1D optical lattice

$$V_{opt}(z) = sE_r \sin^2(qz)$$



Th
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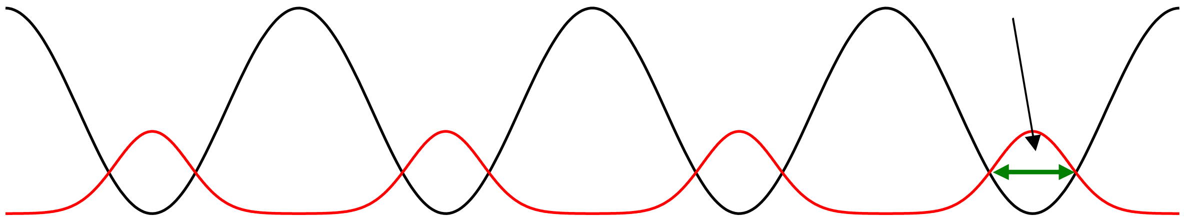
Note

If noninteracting:



BEC in 1D optical lattice

$$V_{opt}(z) = sE_r \sin^2(qz)$$

$$\Psi(z) = \sum_l f(z - ld)$$

Momentum distribution $$n(p) = |\Psi(p)|^2$$

Fourier transform of Wannier function

where $$\Psi(p) = (2\pi\hbar)^{-1/2} \sum_l \int dz\, e^{-ipz/\hbar} f(z - ld) = f_0(p) \sum_l e^{-ildp}$$

If *s*>>1

Number of occupied wells

If noninteracting:

$$n(p) = f_0^2(p) \frac{\sin^2(N_w pd / 2\hbar)}{\sin^2(pd / 2\hbar)}$$

$$f_0(p) = \frac{\sigma^{1/2}}{\pi^{1/4}\hbar^{1/2}} \exp(-p^2\sigma^2 / 2\hbar^2)$$

BEC in 1D optical lattice

$$n(p) = f_0^2(p)\frac{\sin^2(N_w pd/2\hbar)}{\sin^2(pd/2\hbar)}$$

$$f_0(p) = \frac{\sigma^{1/2}}{\pi^{1/4}\hbar^{1/2}}\exp(-p^2\sigma^2/2\hbar^2)$$

The momentum distribution is characterized by series of peaks located at

$$p = 2\pi\hbar n/d = 2n\hbar q_B = 2np_B$$

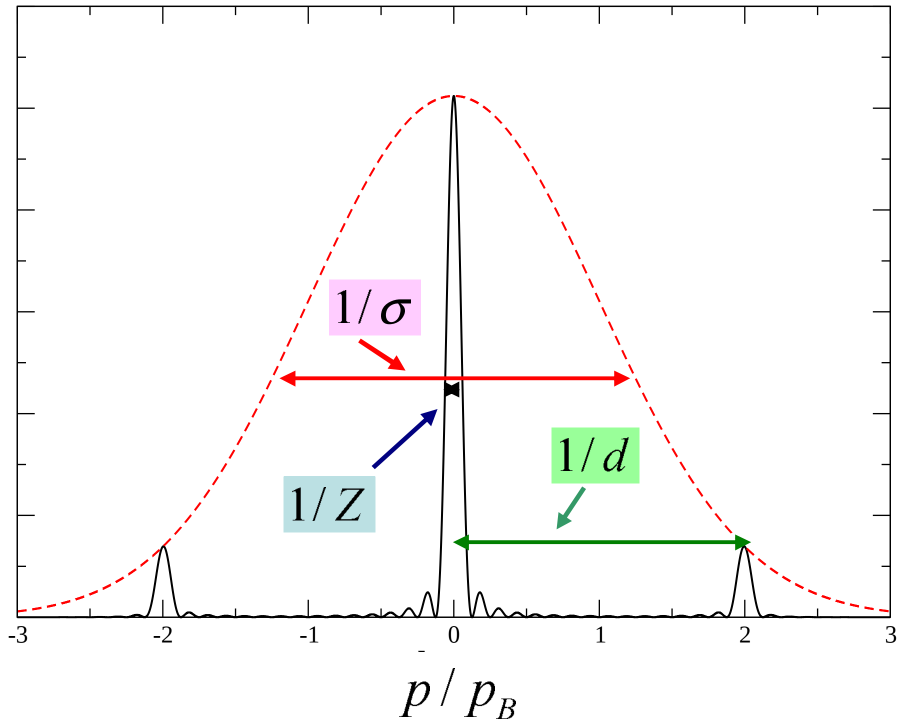
Each peak has relative weight

$$\exp(-4\pi^2 n^2\sigma^2/d^2)$$

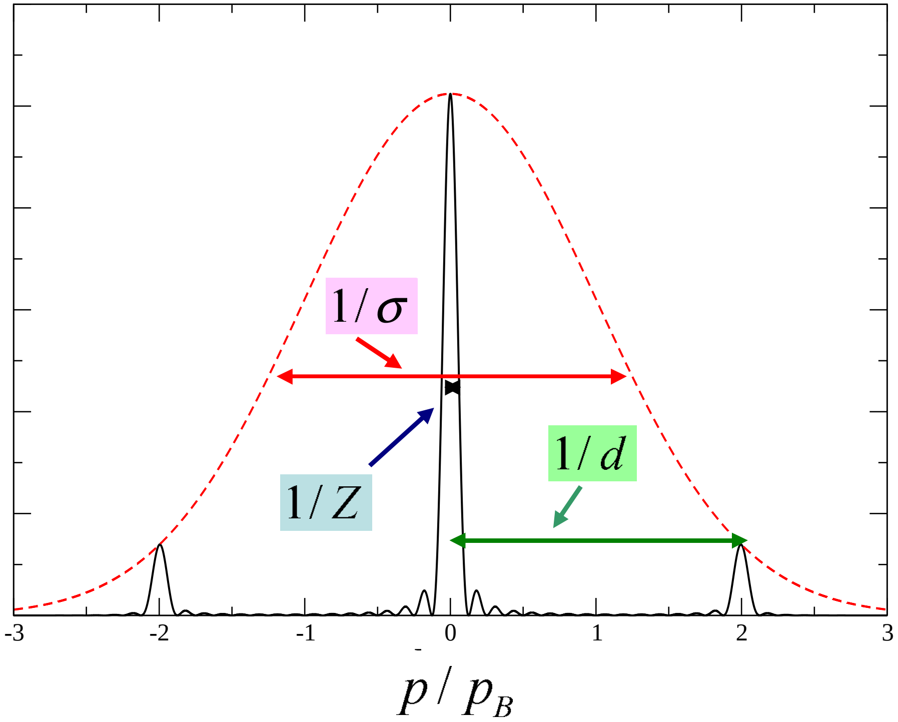
and relative width

$$\hbar/N_w d = \hbar/Z$$

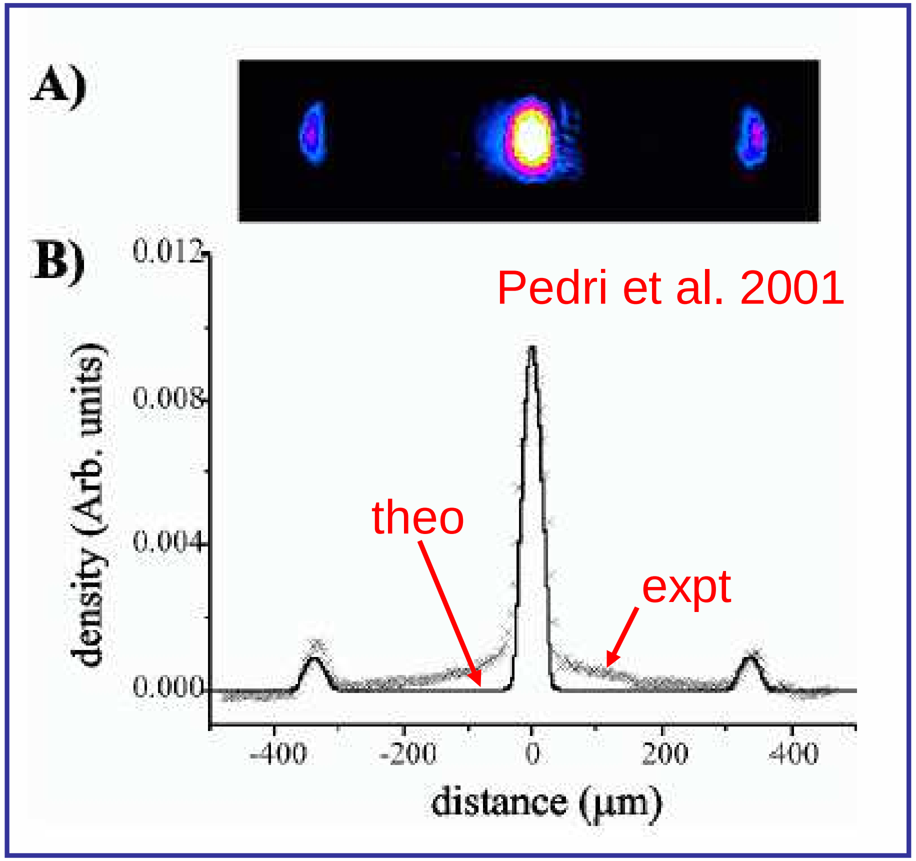
Size of the trapped gas



BEC in 1D optical lattice



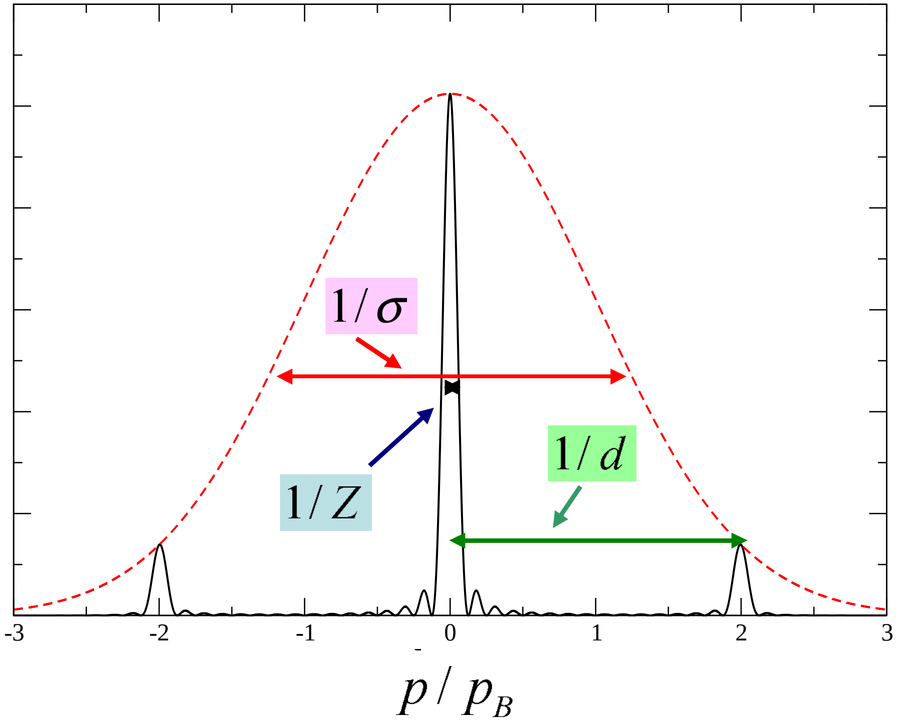
Free expansion of BEC out of a lattice



coherent matter wave diffraction
from a lattice made of **light**
instead of
coherent light diffraction
from a lattice made of **matter**

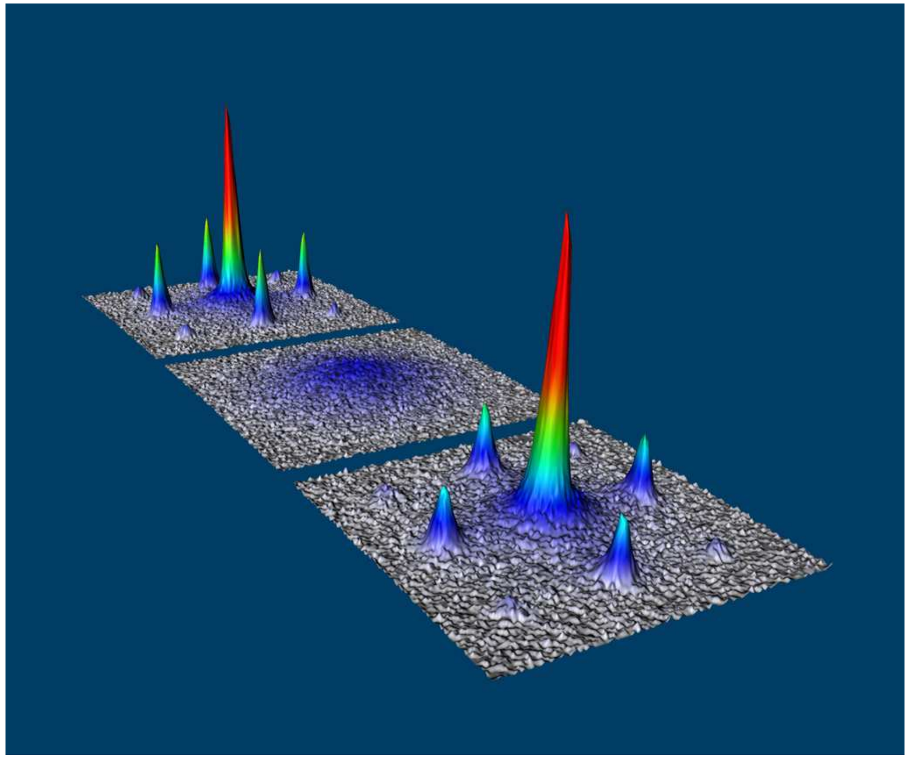
!

BEC in 1D optical lattice



Free expansion of BEC out of a lattice

in 3D (I.Bloch et al., 2002)



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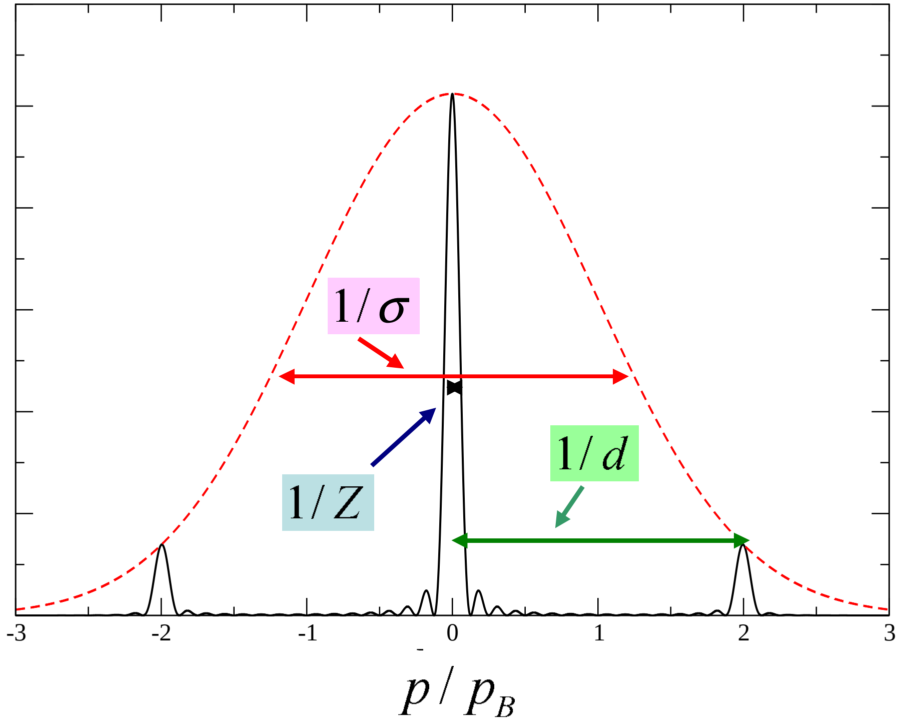
BEC in 1D optical lattice



coherent matter wave diffraction
from a lattice made of **light**
instead of
coherent light diffraction
from a lattice made of **matter**

!

Note:
Interactions and harmonic trapping do not change significantly the mechanism of the expansion and the peak separation provided

$$\mu < E_r$$

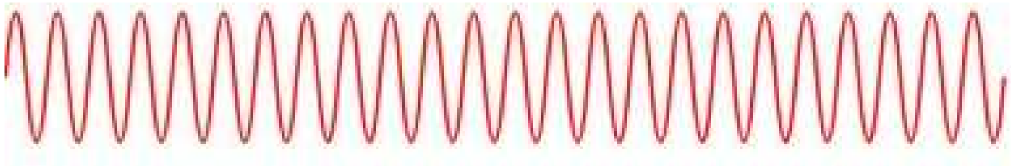
They instead affect the occupation **number of atoms in each well** and hence the **shape** of the **density distribution**.

One can use GP theory (within certain limits of applicability)

Bloch waves and bands

A uniform system is translationally invariant → momentum is a good quantum number

A periodic external potential breaks the translational invariance → momentum is NOT a good quantum number



However, one can always write wavefunctions (and order parameter) in this form:

$$\Psi_P(z) = e^{iPz/\hbar}\varphi_P(z)$$

Bloch waves

(as for electrons in a solid)

where $\varphi_P(z)$ has the same periodicity of the lattice: $\varphi_P(z) = \varphi_P(z+d)$

P is the **quasi-momentum**!

It coincides with true momentum only in the limit $s \to 0$

Bloch waves and bands

For a BEC in an 1D optical lattice, one can use the Bloch wave decomposition

$$\Psi_P(z) = e^{iPz/\hbar}\varphi_P(z)$$

for the order parameter solution of the GP equation.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V_{opt}(z) + g\left|\Psi_P(z)\right|^2\right]\Psi_P(z) = \mu\Psi_P(z)$$

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz} - i\frac{P}{\hbar}\right)^2\varphi_P(z) + \left[g\left|\varphi_P(z)\right|^2 + V_{opt}(z)\right]\varphi_P(z) = \mu(P)\varphi_P(z)$$

P=0 → BEC at rest in the lattice.

P≠0 → BEC moving in the lattice.

Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Since all functions are periodic with period *d*, one can solve this equation in a single lattice site.

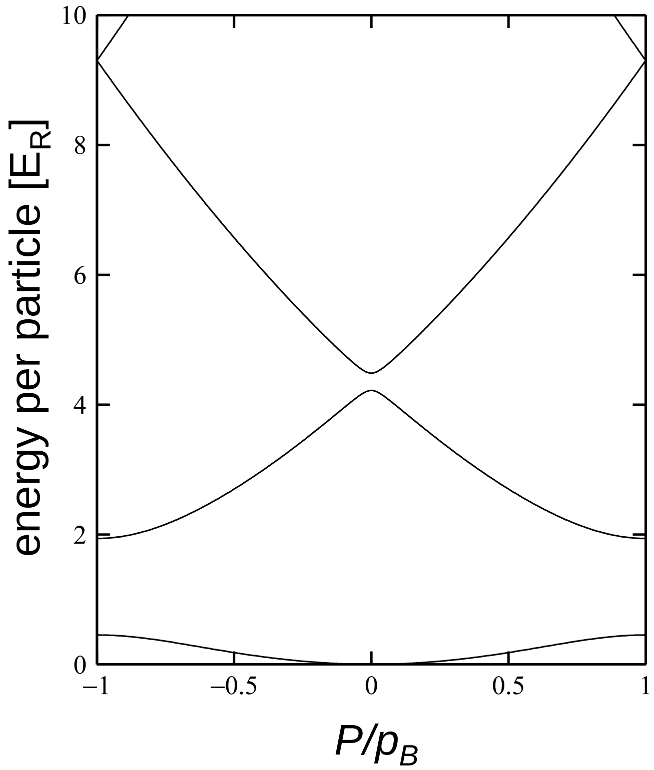
The energy of the solutions will be a function of the quasi-momentum *P*, which can be calculated in the first Brillouin zone:

$$-p_B \leq P \leq p_B\ , \quad p_B = \hbar / d$$

Bloch bands

$$\varepsilon(P)=\frac{E(P)-E(0)}{N}$$

from the solutions of GP eq.



Bloch waves and bands

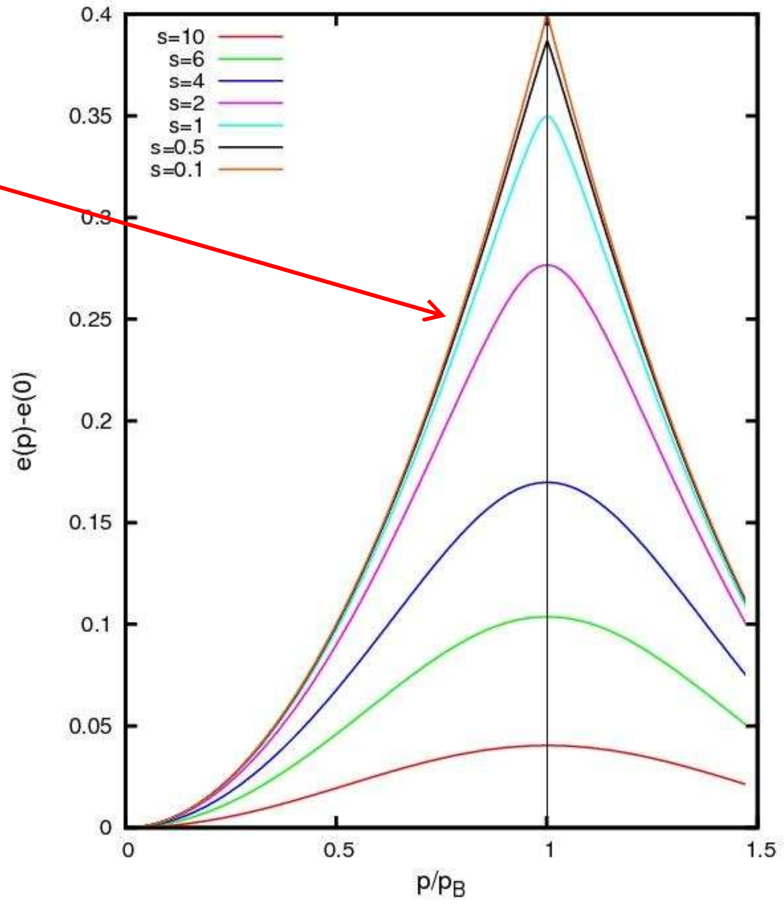
$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Without lattice:

$$\varepsilon_0(P)=P^2/2m$$

average density

Lowest Bloch band ($gn=0.4E_R$)



Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g|\varphi_P(z)|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Without lattice:

$$\varepsilon_0(P)=P^2/2m$$

With lattice, in the small P limit:

$$\varepsilon(P)=P^2/2m^*$$

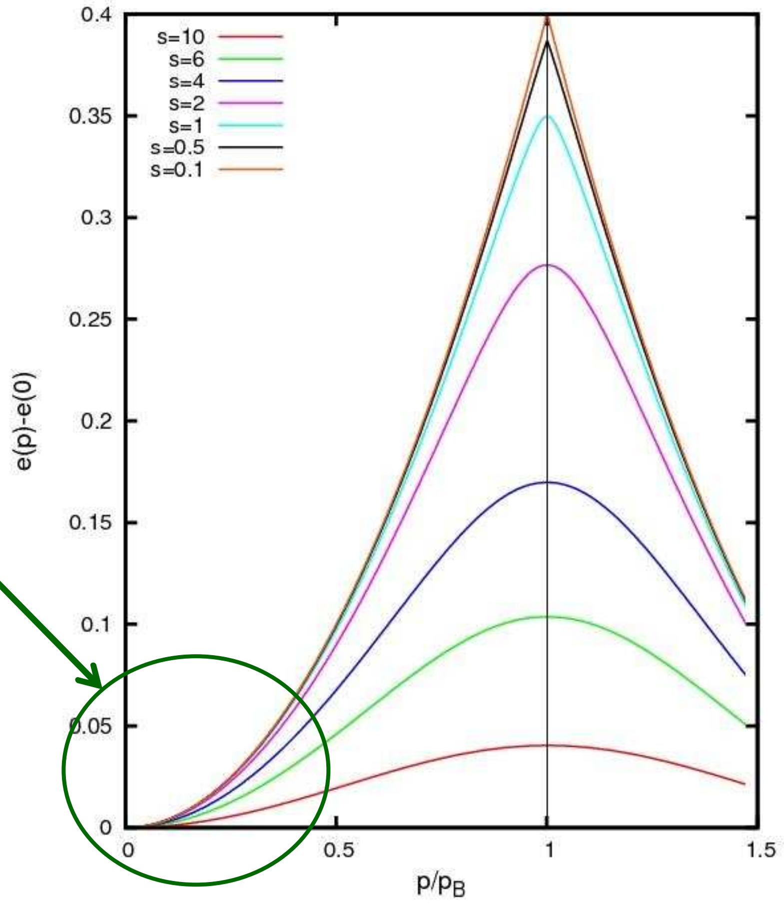
effective mass

At small *P*, BEC flows in the lattice as it were a fluid of particle with mass *m** with current density

$$J=nP/m^*$$

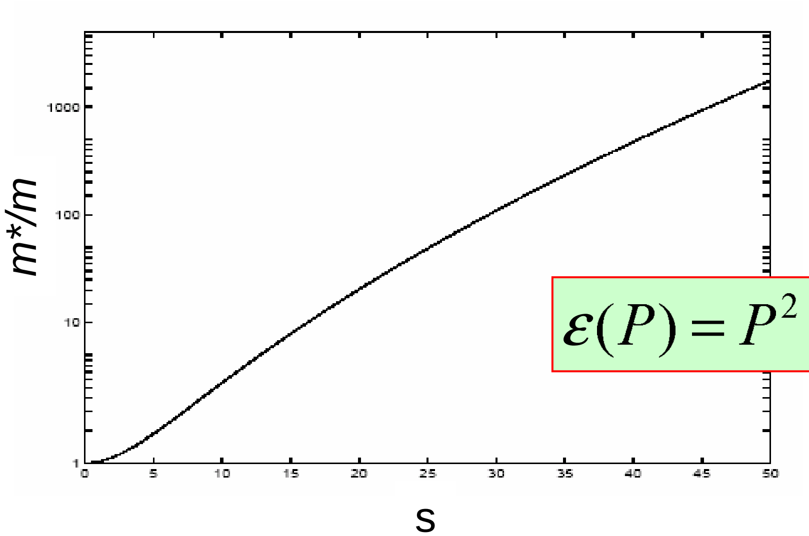
average density

Lowest Bloch band ($gn=0.4E_R$)



Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

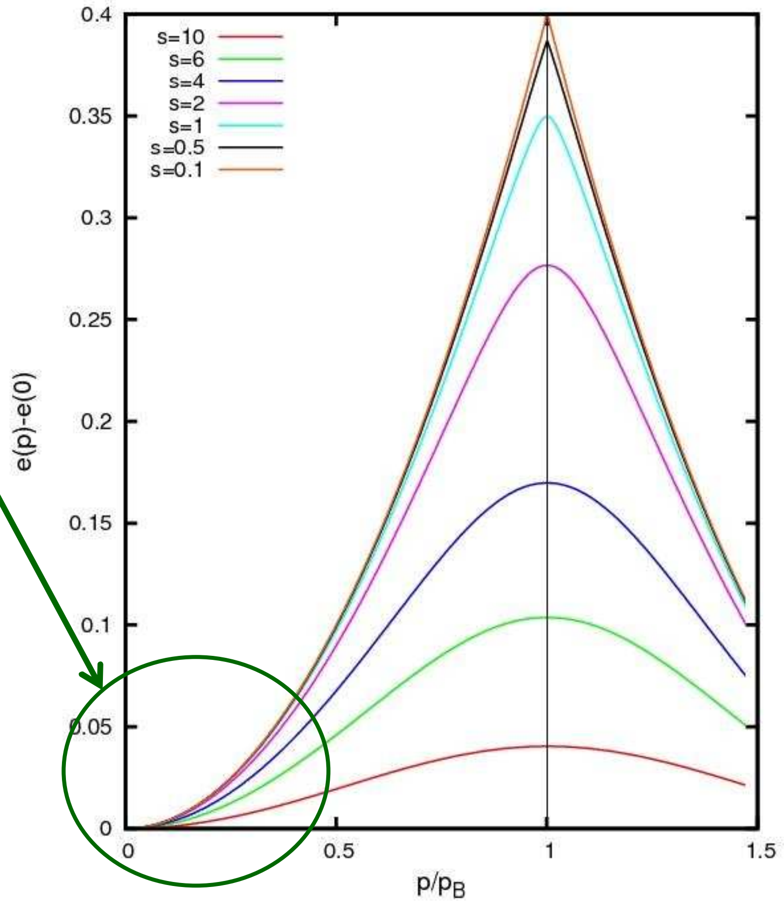


$$\varepsilon(P)=P^2/2m^*$$

*m** increases exponentially with *s*,
for large *s* !
(lattice acts against flow)

average density

Lowest Bloch band ($gn=0.4E_R$)



Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Large s: **tight-binding limit**

$$\varepsilon(P)=2\delta_J\sin^2(Pd/2\hbar)$$

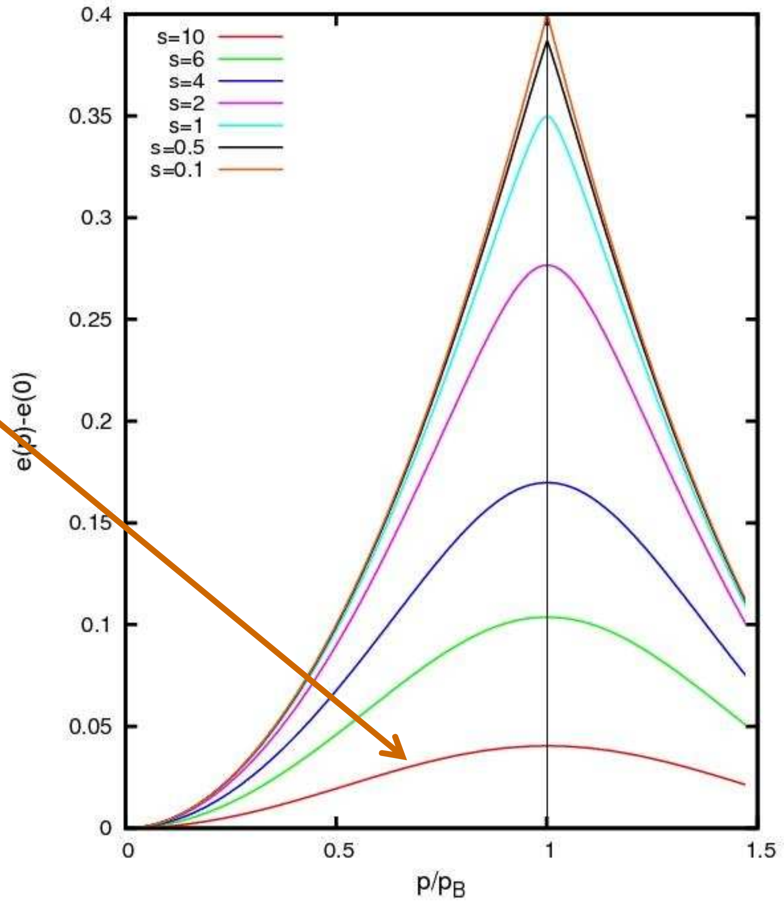
$$\delta_J=\frac{\hbar^2}{m^*d^2}$$

tunnelling energy

In this regime the flow is dominated by macroscopic tunnelling between lattice sites.

average density

Lowest Bloch band ($gn=0.4E_R$)



Bloch waves and bands

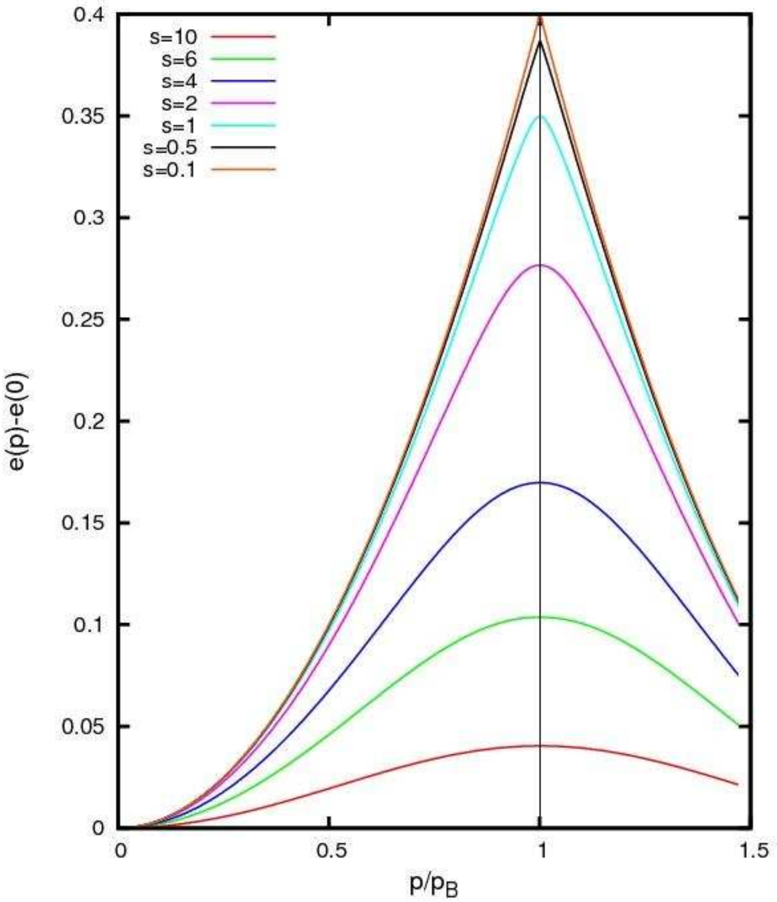
$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g|\varphi_P(z)|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Important remark:

GP equation is nonlinear. Differently from Schrödinger equation, it admits nonlinear stationary states with P larger than p_B.

average density

Lowest Bloch band ($gn=0.4E_R$)



Bloch waves and bands

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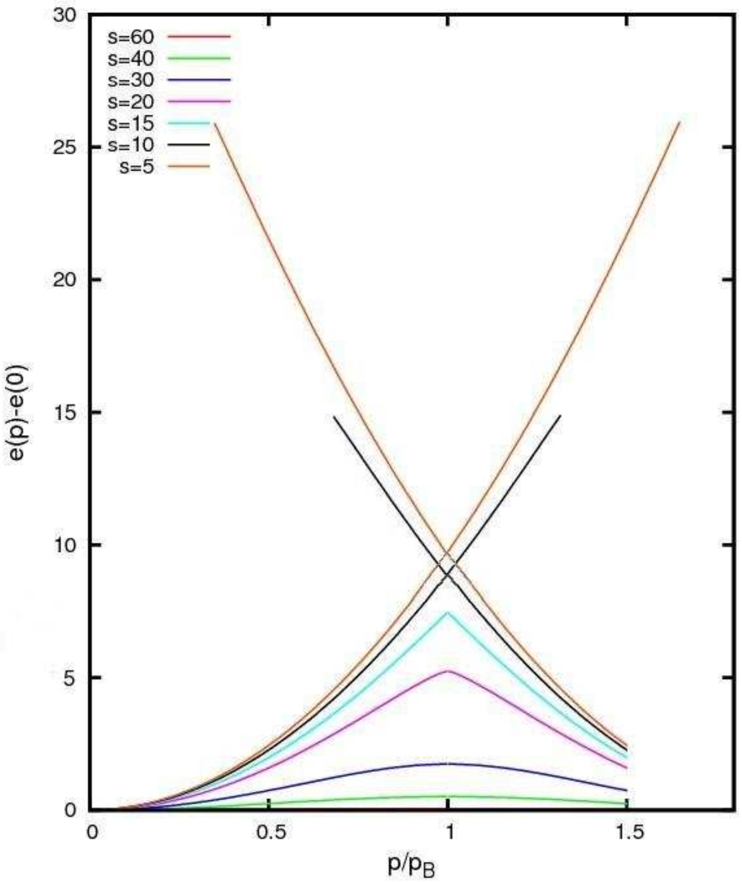
if $2gn > sE_R$

Swallaw tails

Lowest Bloch band ($gn=10E_R$)



Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Important remark:

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if $2gn > sE_R$ ⇩

Swallaw tails

Machholm et al., PRA 67, 053613 (2003).

PHYSICAL REVIEW A **67**, 053613 (2003)

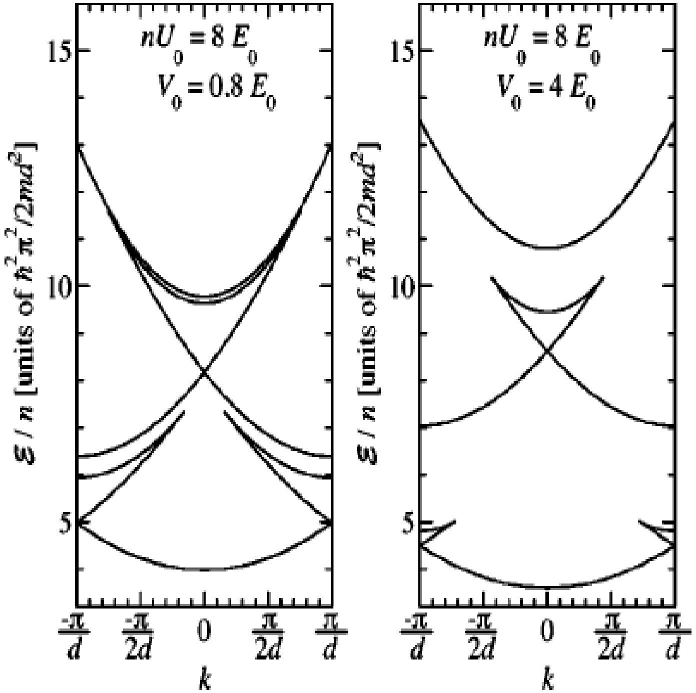


FIG. 1. Energy per particle as a function of wave number for the owest bands. The results are obtained from numerical calculations ›ased on wave function (6), as described in Sec. II B.

Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Important remark:
GP equation is nonlinear. Differently from Schrödinger equation, it admits nonlinear stationary states with P larger than p_B.

if $2gn > sE_R$

⇓

Swallaw tails

Other effects of nonlinearity:
Solitons (bright, dark, gray,…)

Excitations of a BEC in a lattice

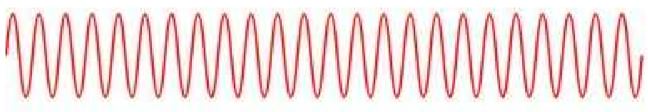
Excitations in the linear (small amplitude) regime: Bogoliubov quasiparticles.

$$\hbar\omega_j u_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)u_j + g\Psi_0^2 v_j$$

$$-\hbar\omega_j v_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)v_j + g\Psi_0^{*2} u_j$$

with

$$\Psi_0(\mathbf{r},t) = e^{-i\mu t}[\,\Psi_0(\mathbf{r}) + u_j(\mathbf{r})e^{-i\omega_j t} + v_j^*(\mathbf{r})e^{i\omega_j t}\,] = e^{-i\mu t}[\,\Psi_0(\mathbf{r}) + \delta\Psi_0(\mathbf{r})]$$



In a periodic potential → Bloch waves

$$\Psi_0(z,t) = e^{-i\mu(P)t}\, e^{iPz/\hbar}\,[\,\varphi_P(z) + \delta\varphi_P(z,t)\,]$$

with $\delta\varphi_P(z,t) = \sum_{q,j}[\,u_{qP,j}(z)e^{i(qz-\omega_{qP,j}t)} + v^*_{qP,j}(z)e^{-i(qz-\omega_{qP,j}t)}\,]$

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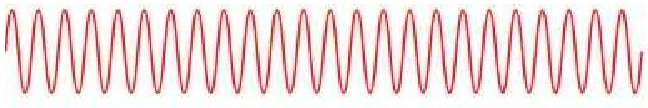

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with $\delta\varphi_P(z,t) = \sum_{q,j}[u_{qP,j}(z)e^{i(qz-\omega_{qP,j}t)} + v^*_{qP,j}(z)e^{-i(qz-\omega_{qP,j}t)}]$

- P: quasi-momentum of the condensate
- q: quasi-momentum of the quasiparticle
- j: band index

Excitations of a BEC in a lattice

$$\hbar\omega_j u_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)u_j + g\Psi_0^2 v_j$$

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Excitations of a BEC in a lattice

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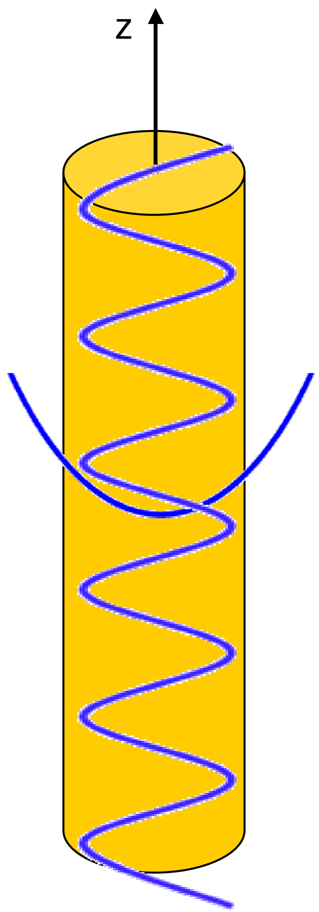


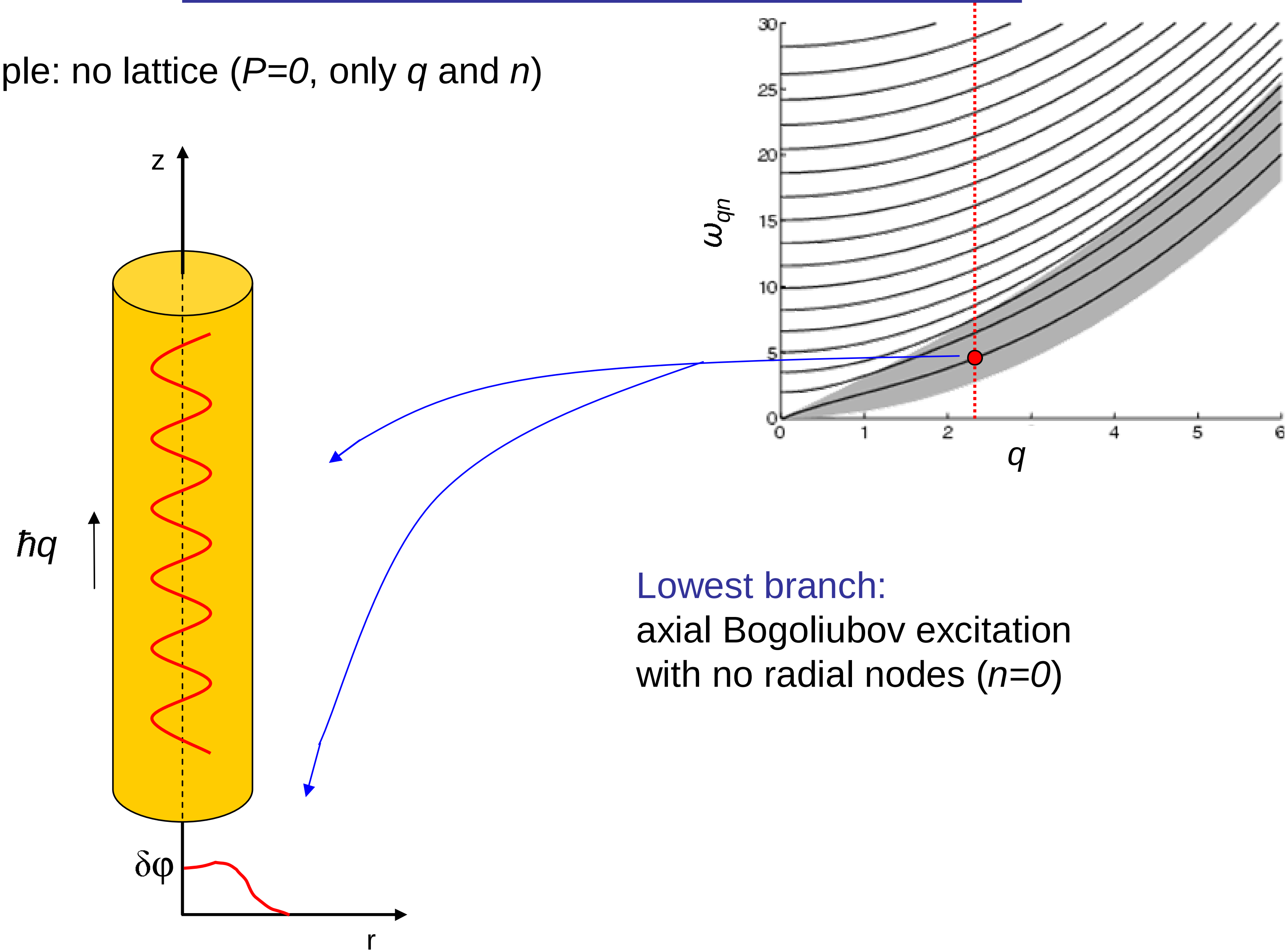
Including transverse radial trapping:

$$\delta\varphi_P(r,z,t) = \sum_{q,j,n} [u_{qP,j,n}(z) e^{i(qz-\omega_{qP,j,n}t)} + v^*_{qP,j,n}(z) e^{-i(qz-\omega_{qP,j,n}t)}]$$

radial quantum number (number of radial nodes)

Excitations of a BEC in a lattice

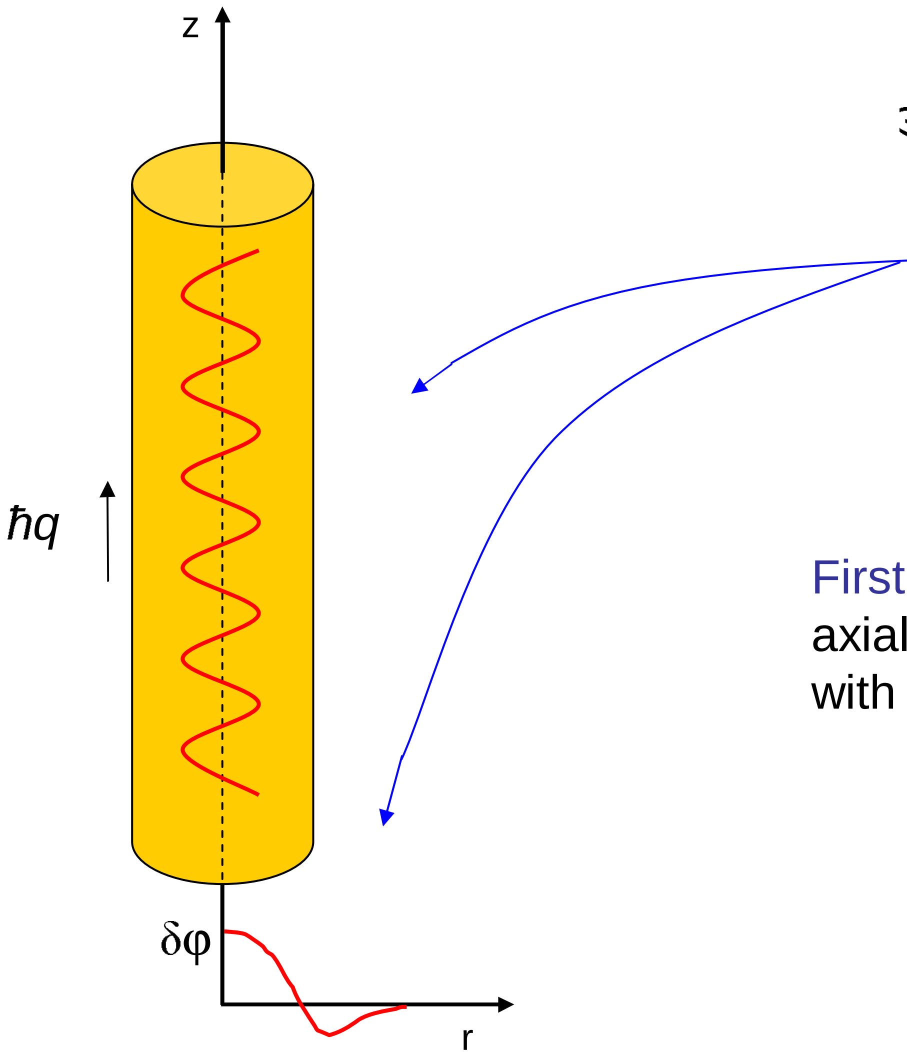
Example: no lattice (*P=0*, only *q* and *n*)

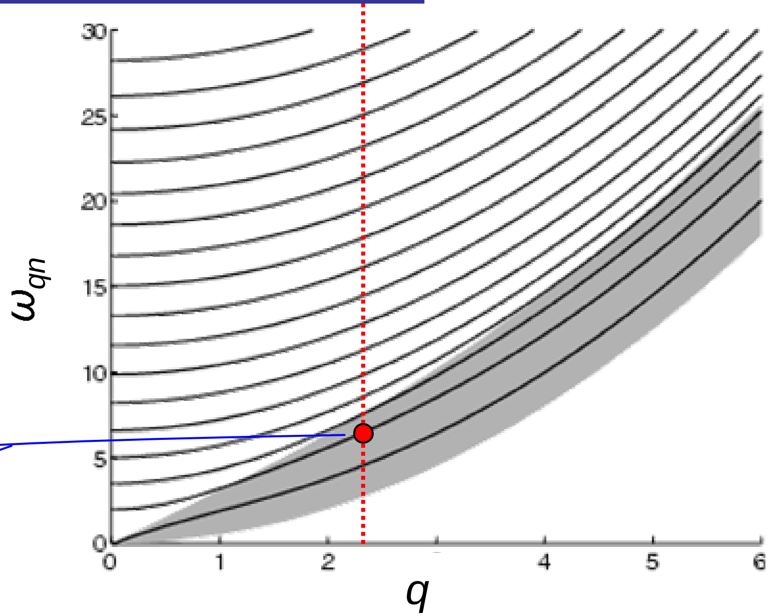


Lowest branch:
axial Bogoliubov excitation
with no radial nodes (*n=0*)

Excitations of a BEC in a lattice

Example: no lattice (*P=0*, only *q* and *n*)

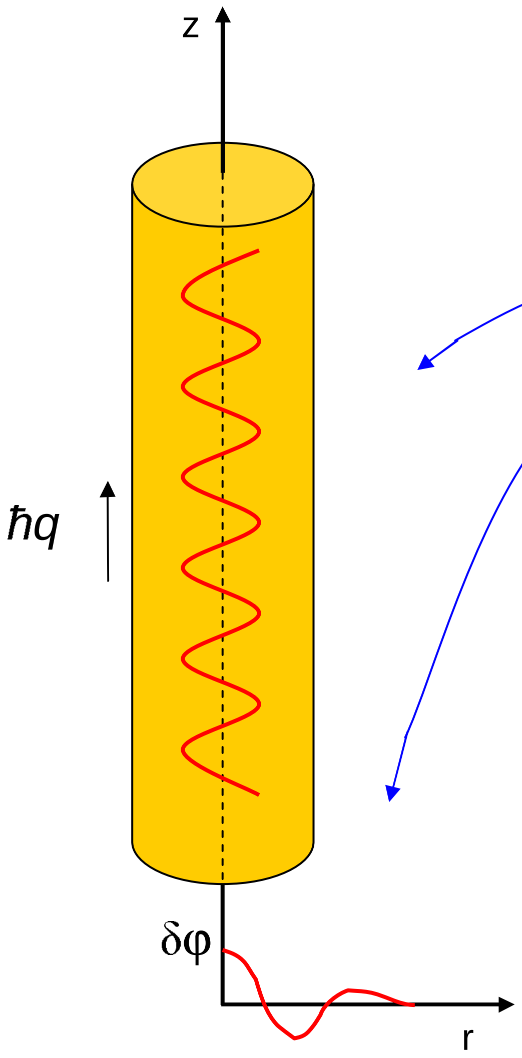


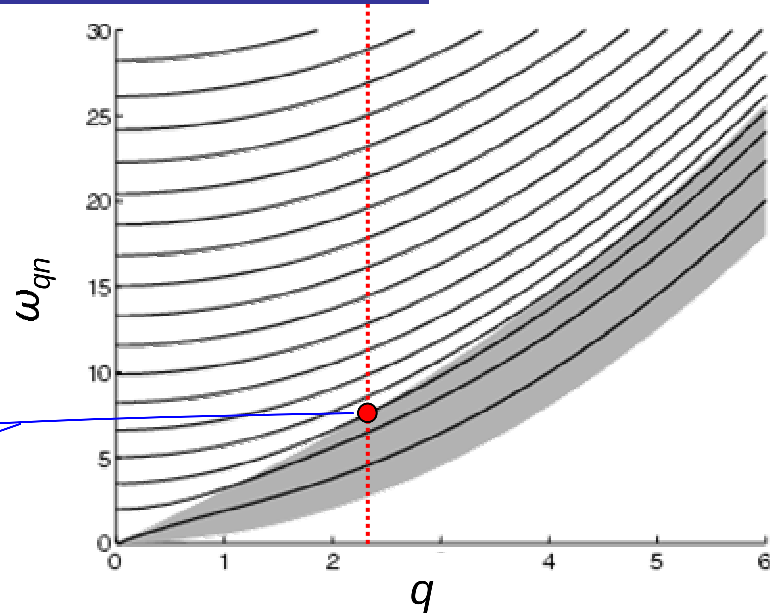


First excited radial branch:
axial Bogoliubov excitation
with one radial node.

Excitations of a BEC in a lattice

Example: no lattice (*P=0*, only *q* and *n*)



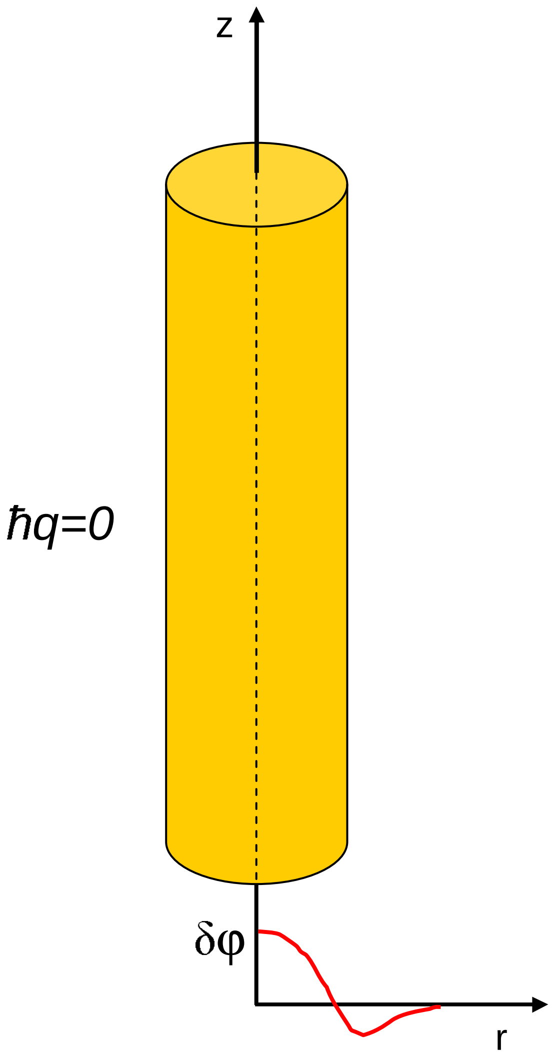


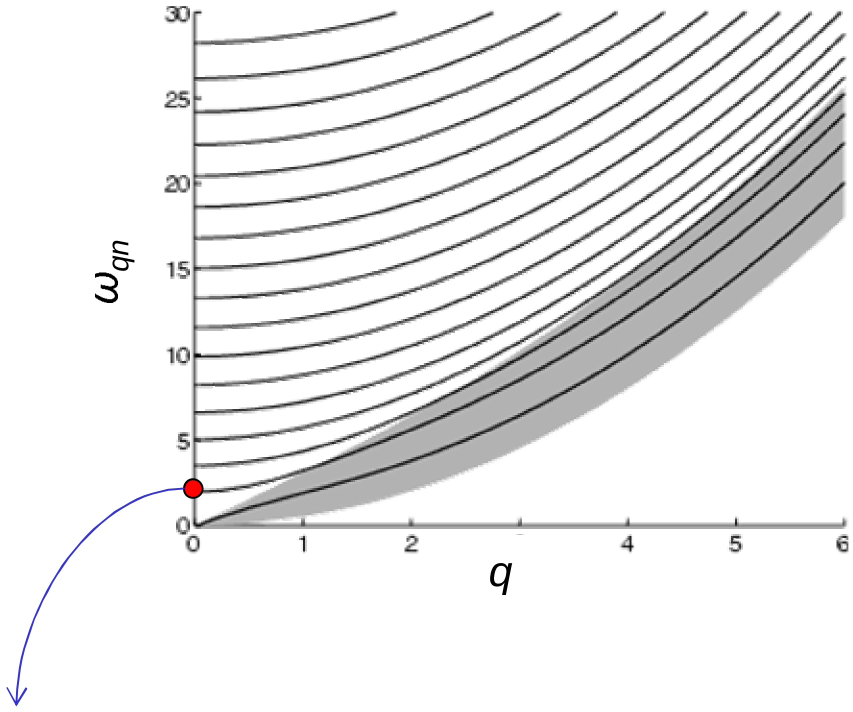
Second excited radial branch:
axial Bogoliubov excitation
with two radial nodes.

... and so on

Excitations of a BEC in a lattice

Example: no lattice (*P=0*, only *q* and *n*)

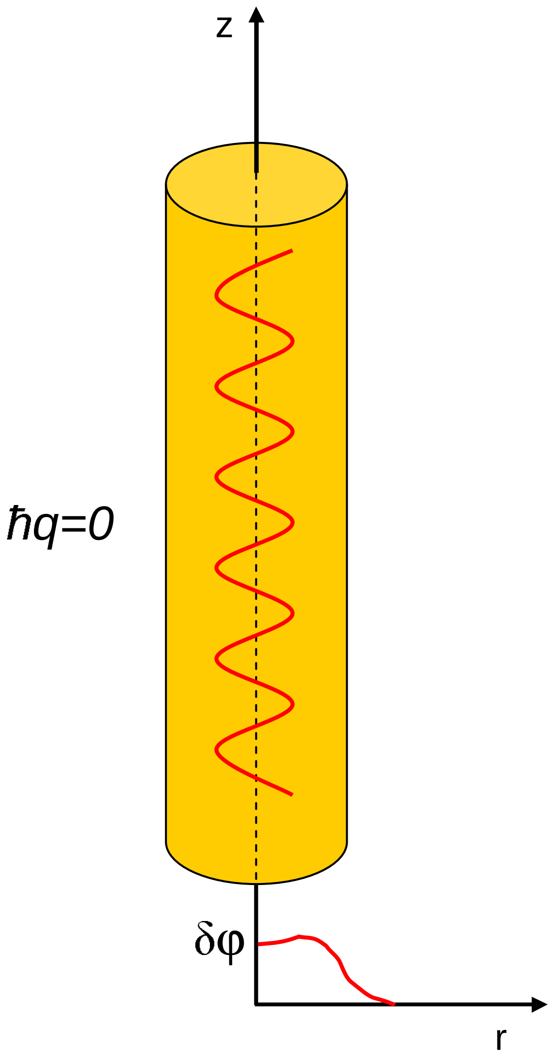


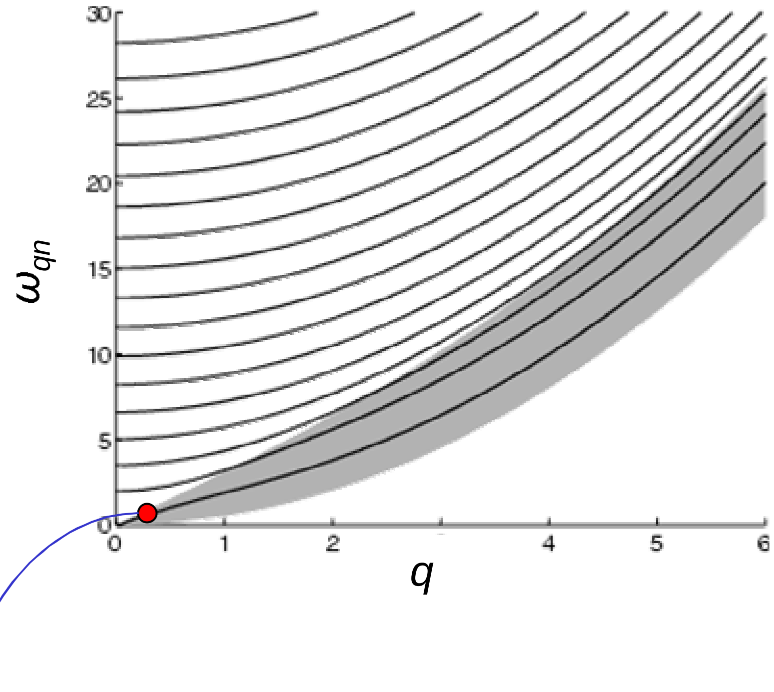


Radial breathing mode:
purely compressional excitation
$\omega=2\omega_{trap}$

Excitations of a BEC in a lattice

Example: no lattice (*P=0*, only *q* and *n*)

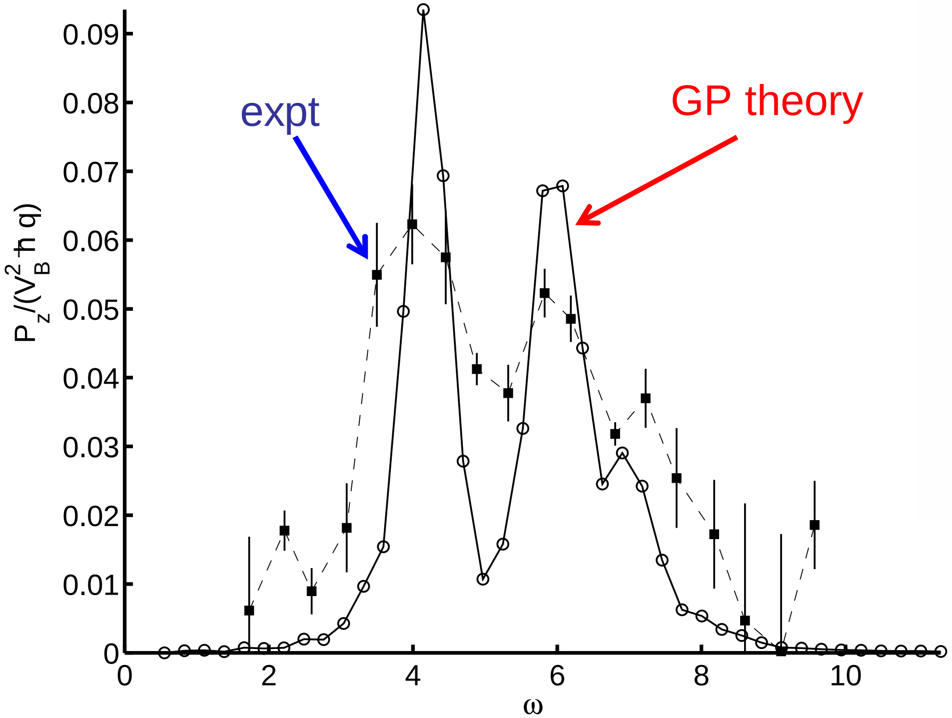




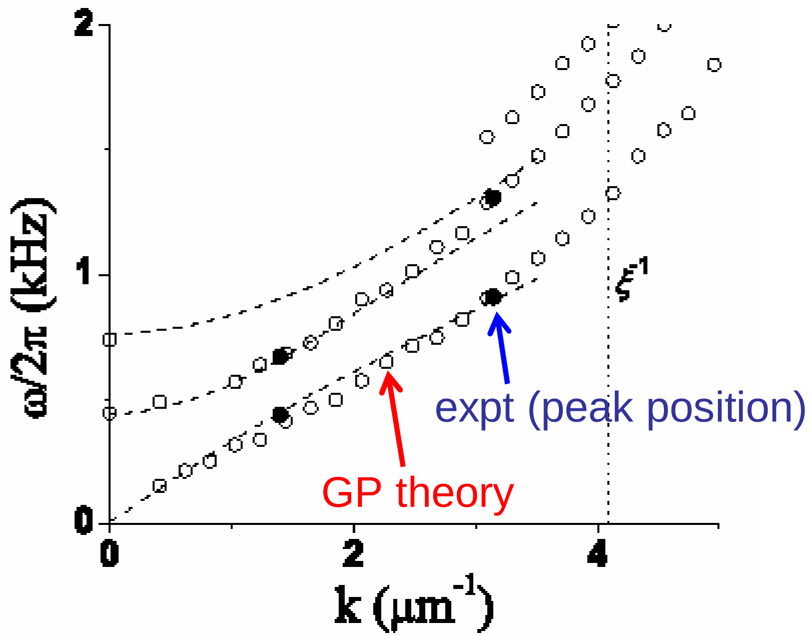
Longitudinal Bogoliubov phonon:
small q ($q \ll \xi^{-1}$), long wavelength.
$\omega = cq$, with c sound velocity.

Excitations of a BEC in a lattice

Spectroscopic measurements by means of light (**Bragg**) scattering. Measure of the total momentum transferred to a BEC. Resonant response at the Bogoliubov frequencies.



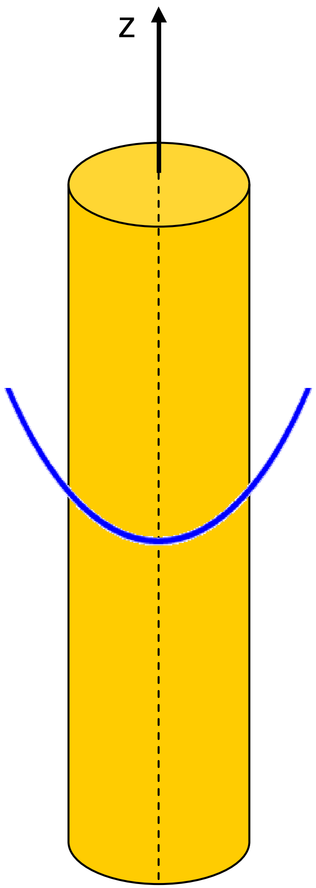
Multi-branch Bogoliubov spectrum

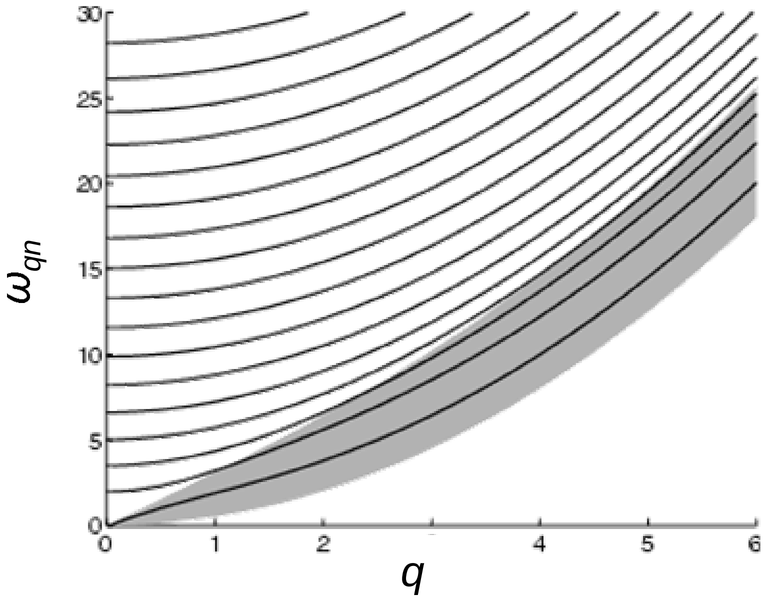


J.Steinhauer, N.Katz, R.Ozeri, N.Davidson, C.Tozzo, F.Dalfovo, PRL **90**, 060404 (2003)

Excitations of a BEC in a lattice

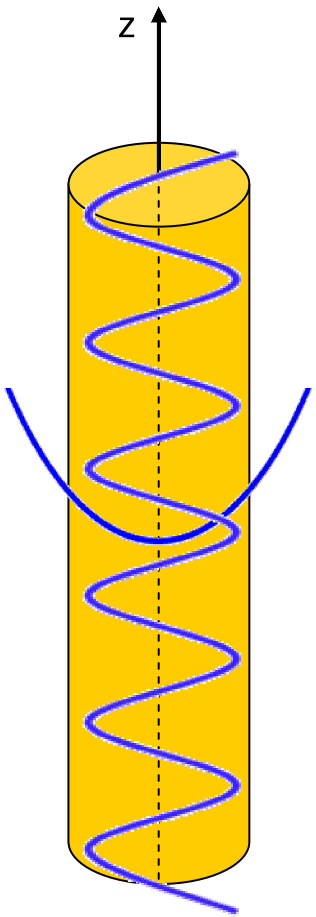
Example: no lattice (*P=0*, no *j*, only *q* and *n*)

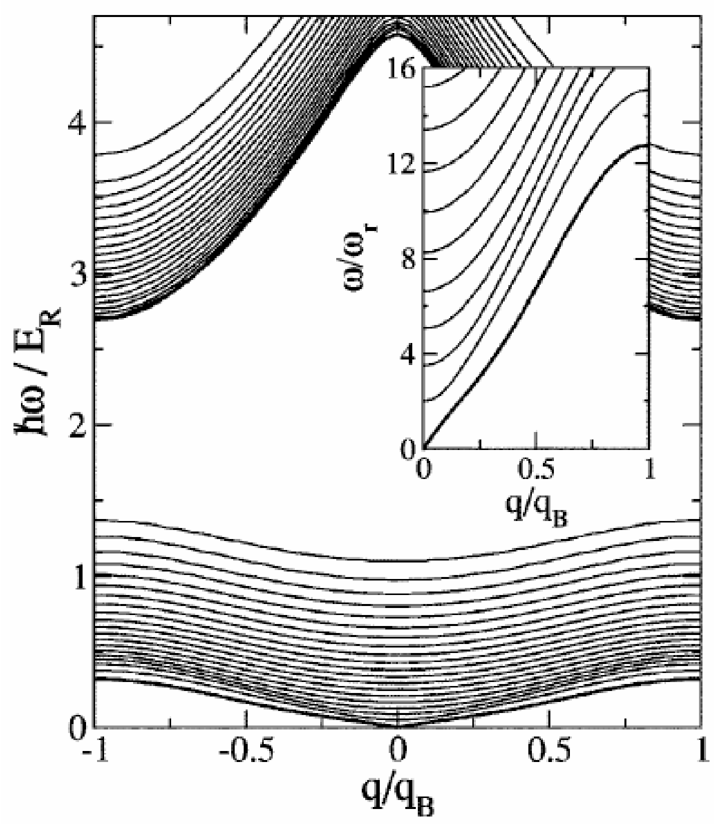




Excitations of a BEC in a lattice

In a lattice + transverse trap

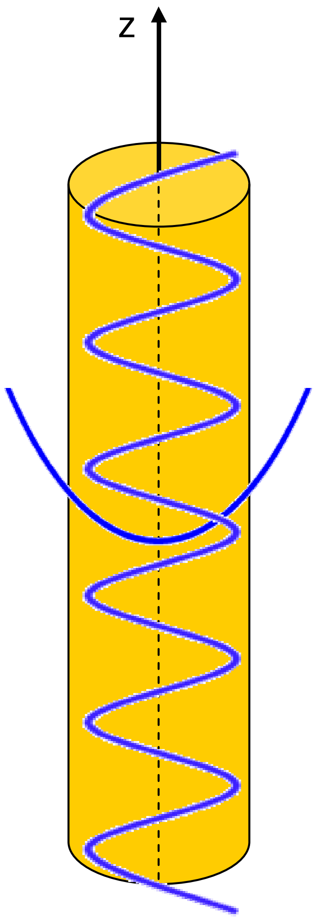




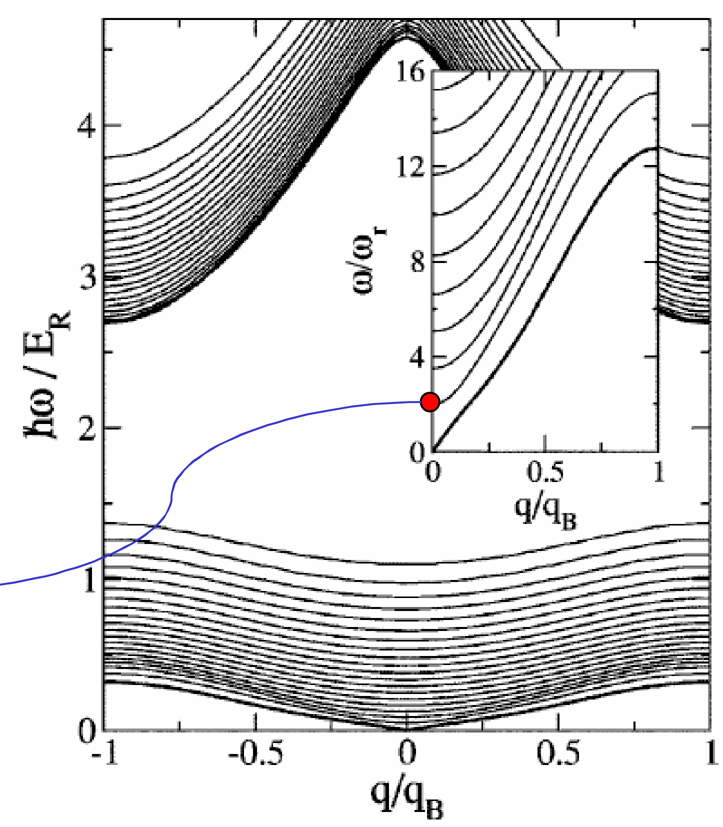
Excitation spectrum of a BEC at rest (**P=0**) in a lattice with s=5. Lowest two Bloch bands, 20 radial branches.

Excitations of a BEC in a lattice

In a lattice + transverse trap



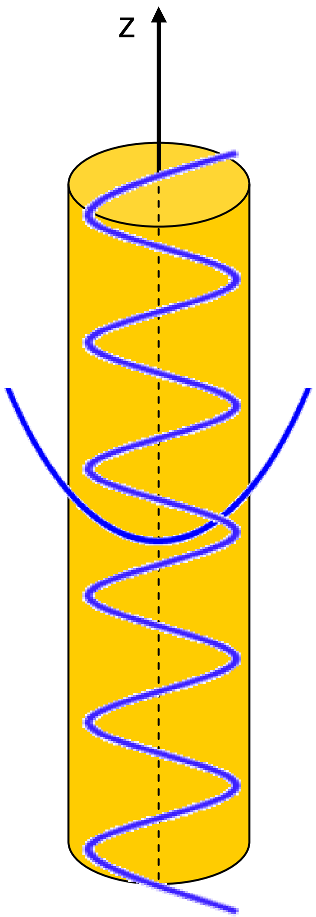
radial breathing
$\omega=2\omega_{trap}$



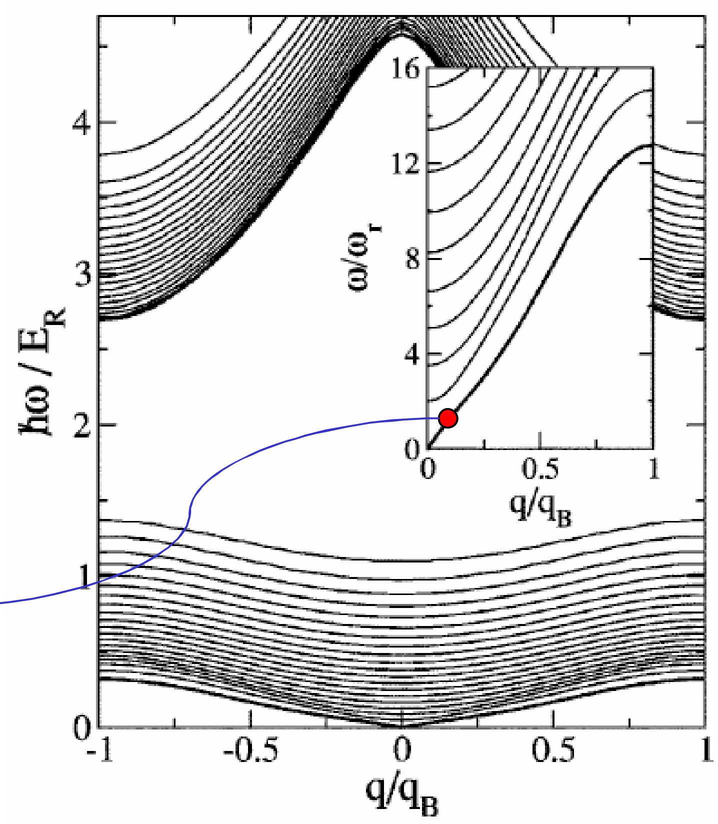
Excitation spectrum of a BEC at rest (P=0) in a lattice with s=5. Lowest two Bloch bands, 20 radial branches.

Excitations of a BEC in a lattice

In a lattice + transverse trap



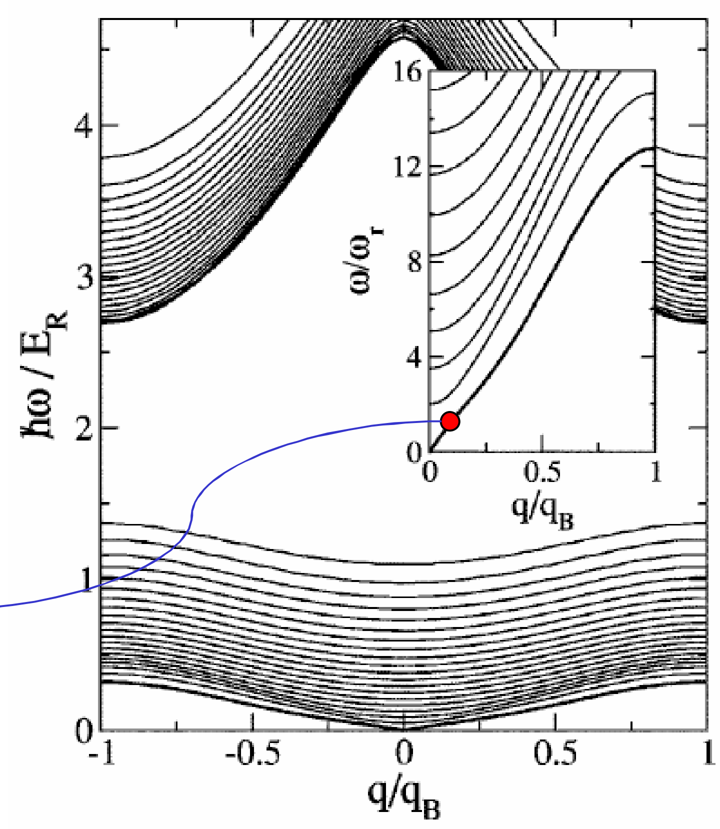
longitudinal
phonon
ω=cq



Excitation spectrum of a BEC at rest (P=0) in a lattice with s=5. Lowest two Bloch bands, 20 radial branches.

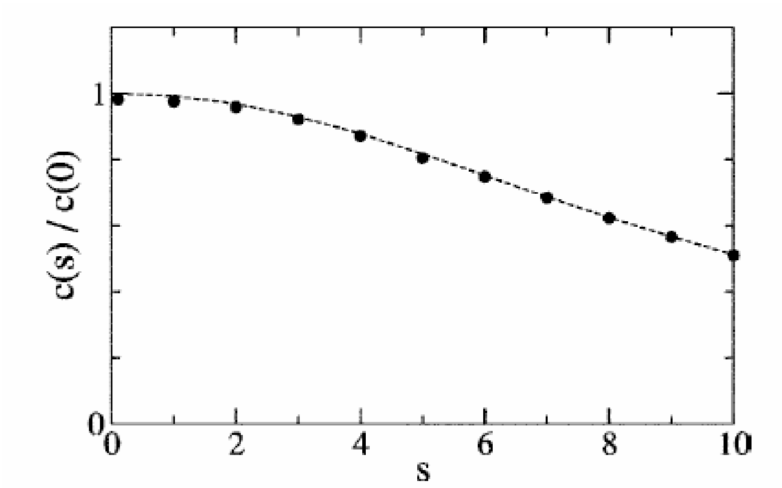
Excitations of a BEC in a lattice

In a lattice + transverse trap



Bogoliubov sound velocity of the lowest phononic branch *vs.* the analytic prediction $c=(\kappa m^*)^{-1/2}$

longitudinal phonon
$\omega = cq$



Excitation spectrum of a BEC at rest (P=0) in a lattice with s=5. Lowest two Bloch bands, 20 radial branches.

Excitations of a BEC in a lattice

In a lattice + transverse trap

P≠0 → BEC moving in the lattice

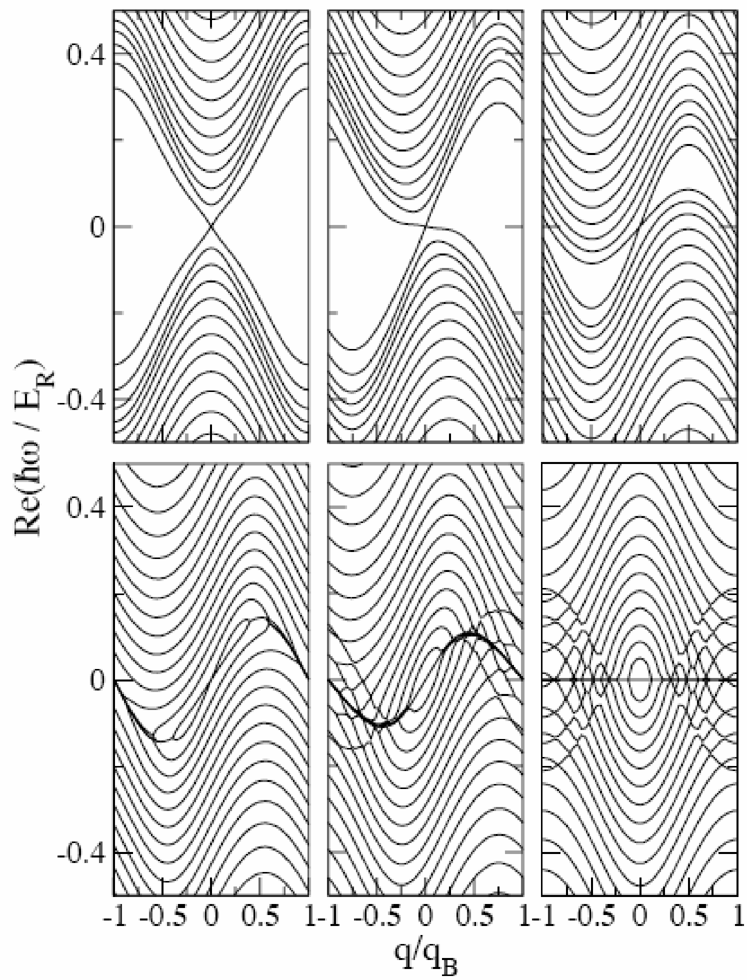
The Bogoliubov equations give the excitations on top of the moving BEC

Remember:

P : quasi-momentum of the condensate

ħq : quasi-momentum of the excitation

$q_B=\pi/d$: Bragg wavevector

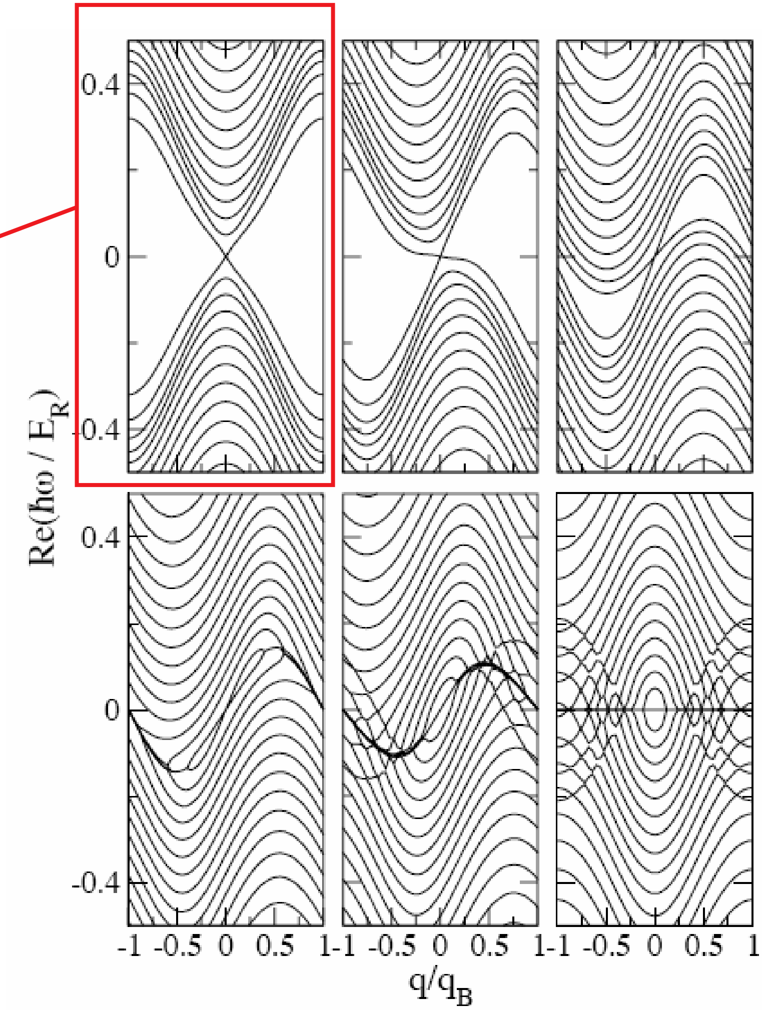


Real part of the excitation spectrum for P=0,0.25,0.5,0.55,0.75,1 p_B.
Lowest band only.

Excitations of a BEC in a lattice

In a lattice + transverse trap

P=0
BEC at rest



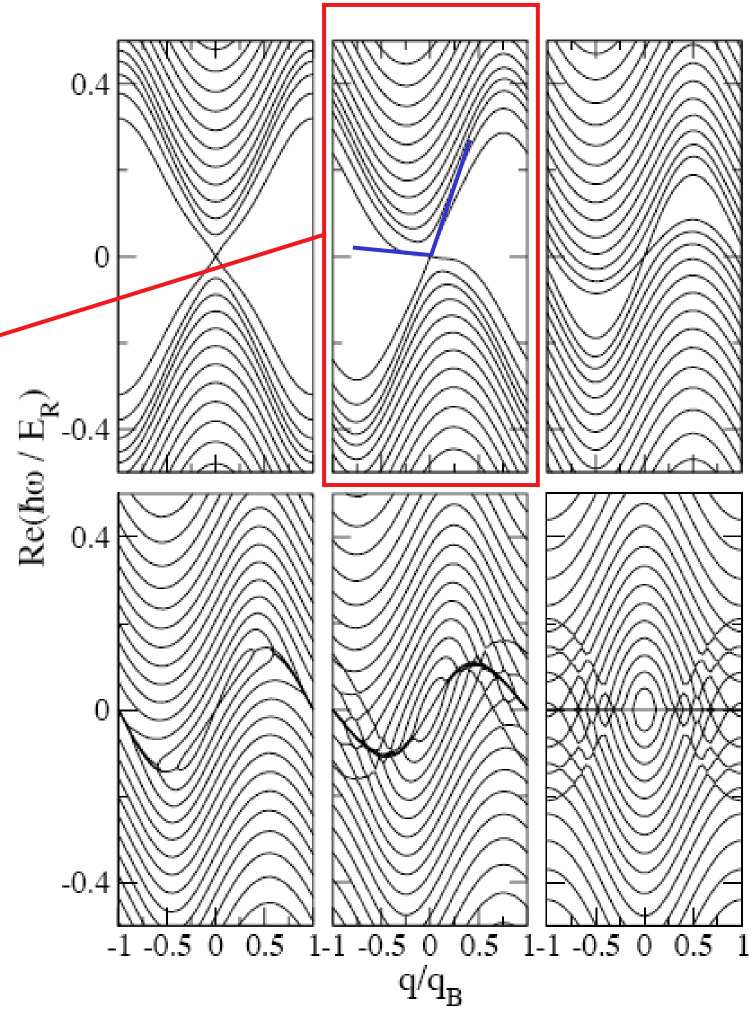
Real part of the excitation spectrum for *P*=0,0.25,0.5,0.55,0.75,1 p_B.
Lowest band only.

Excitations of a BEC in a lattice

In a lattice + transverse trap

P≠*0*
BEC moving

Note: Doppler effect on sound speed measured in the lattice frame.



Real part of the excitation spectrum for P=0,0.25,0.5,0.55,0.75,1 $p_{B.}$
Lowest band only.

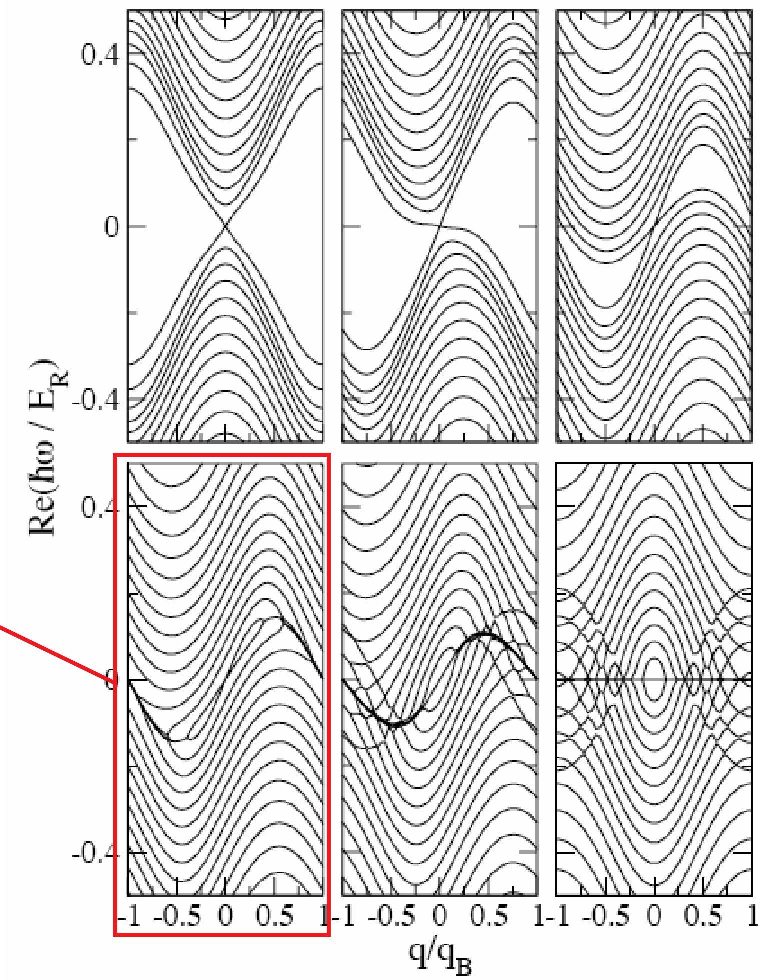
Excitations of a BEC in a lattice

In a lattice + transverse trap

Coupling between propagating and counter-propagating excitations.

Complex eigenfrequencies

Dynamical instability



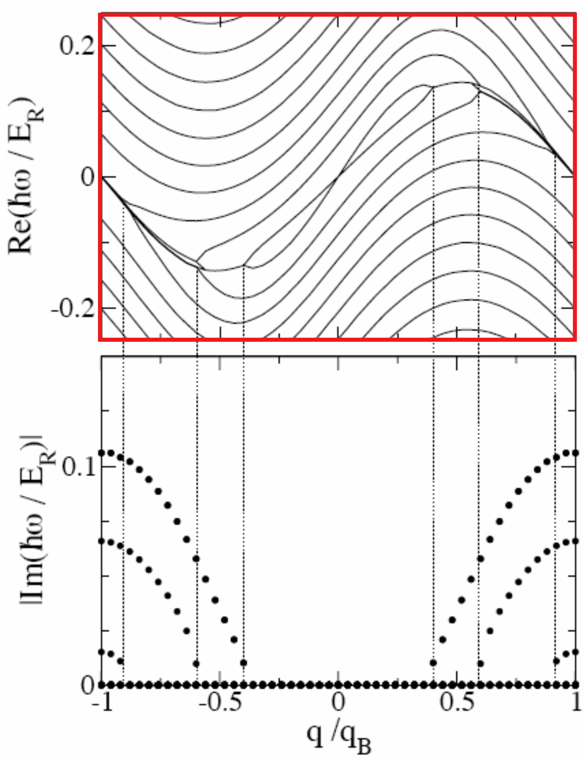
Real part of the excitation spectrum for *P*=0,0.25,0.5,0.55,0.75,1 p_B.
Lowest band only.

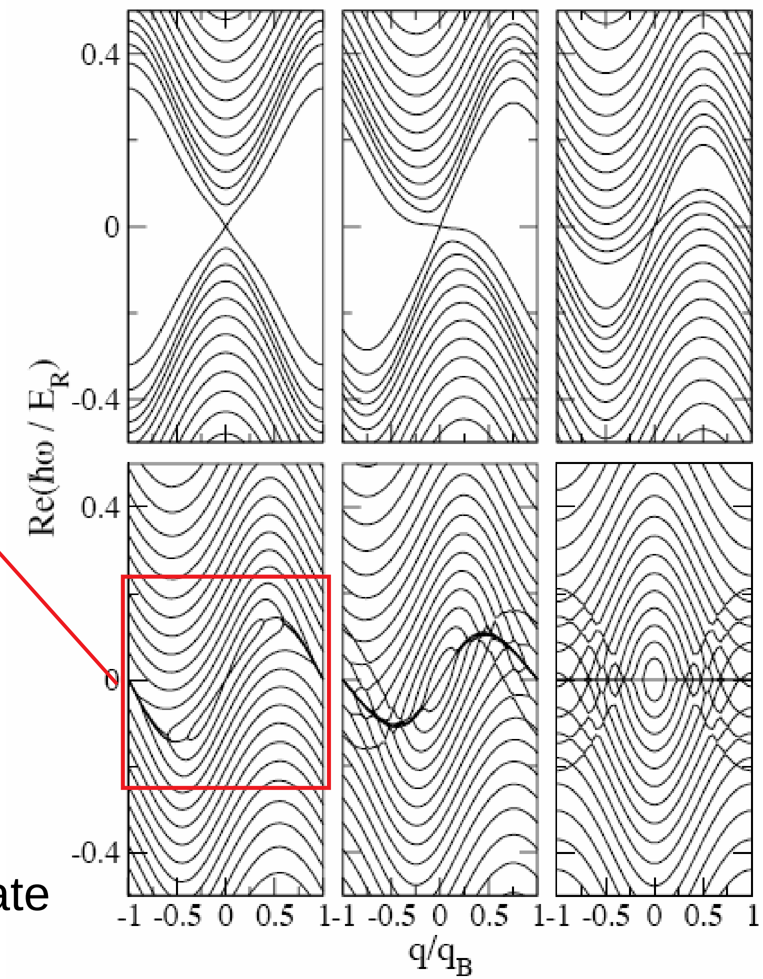
Excitations of a BEC in a lattice

In a lattice + transverse trap

real

imaginary





Phonon-antiphonon resonance ⇒ a conjugate pair of complex frequencies appears.

⇒ resonance condition for two particles decaying into two different Bloch states

Real part of the excitation spectrum for P=0,0.25,0.5,0.55,0.75,1 p_B.
Lowest band only.

Energetic and dynamical instability

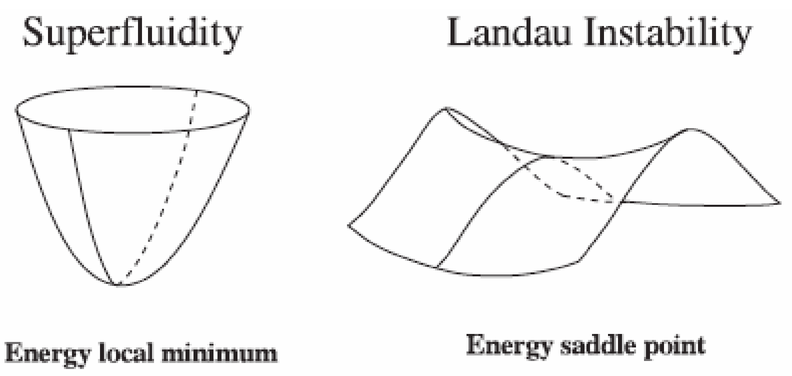
- Stationary solution + fluctuations:

$$\psi = \psi_0 + \delta\psi$$

$$\delta E = \int (\delta\psi^* \delta\psi)\, M(p) \begin{pmatrix} \delta\psi \\ \delta\psi^* \end{pmatrix}$$

$$M(p) = \begin{pmatrix} H_0 + 2g|\psi_0|^2 & g\psi_0^2 \\ g\psi_0^{*2} & H_0 + 2g|\psi_0|^2 \end{pmatrix}$$

- Negative eigenvalues of M(p) ⇒ energetic (Landau) instability. It takes place in the presence of dissipation (impurities, obstacles, thermal excitations, etc.)



- Time dependent fluctuations:

$$\psi(t) = \psi_0 + \delta\psi(t)$$

- Bogoliubov equations:

$$i\hbar\partial_t \begin{pmatrix} \delta\psi \\ \delta\psi^* \end{pmatrix} = \sigma_z M(p) \begin{pmatrix} \delta\psi \\ \delta\psi^* \end{pmatrix}$$

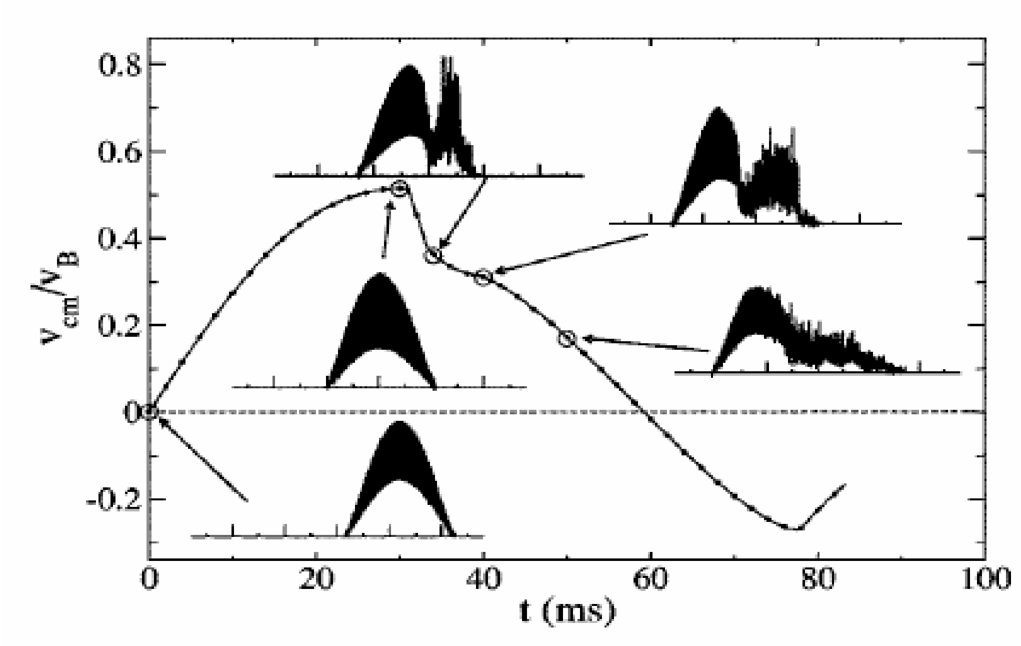
$$\sigma_z \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

- Imaginary eigenvalues of M(p) ⇒ modes that grow exponentially with time. Dynamical instability.

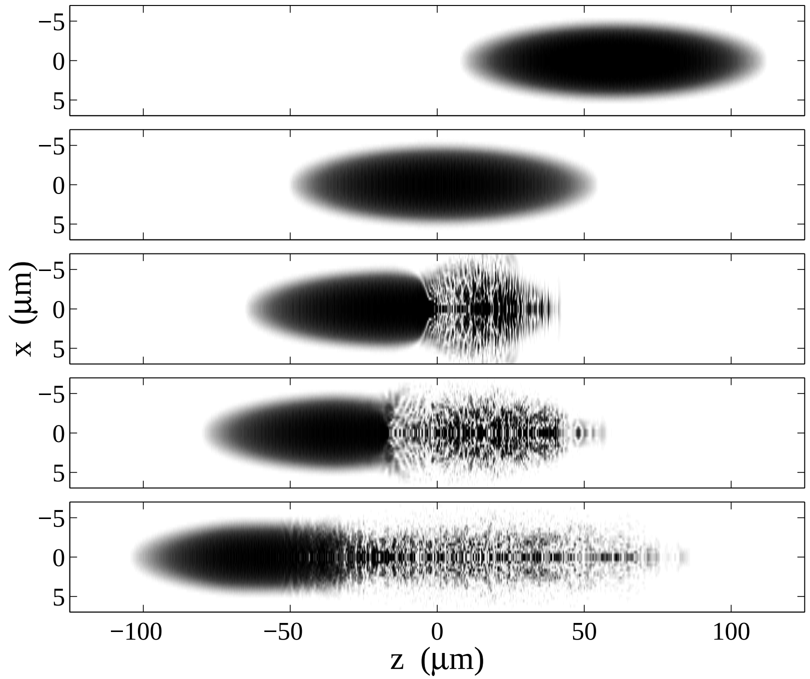
Excitations of a BEC in a lattice

From Bogoliubov equation: **stability diagram** in the (P,q) plane at a given s.

The results agree with time-dependent GP **simulations** and with **experiments** (at LENS, Florence) on the disruption of superfluidity of a BEC accelerated in a lattice.

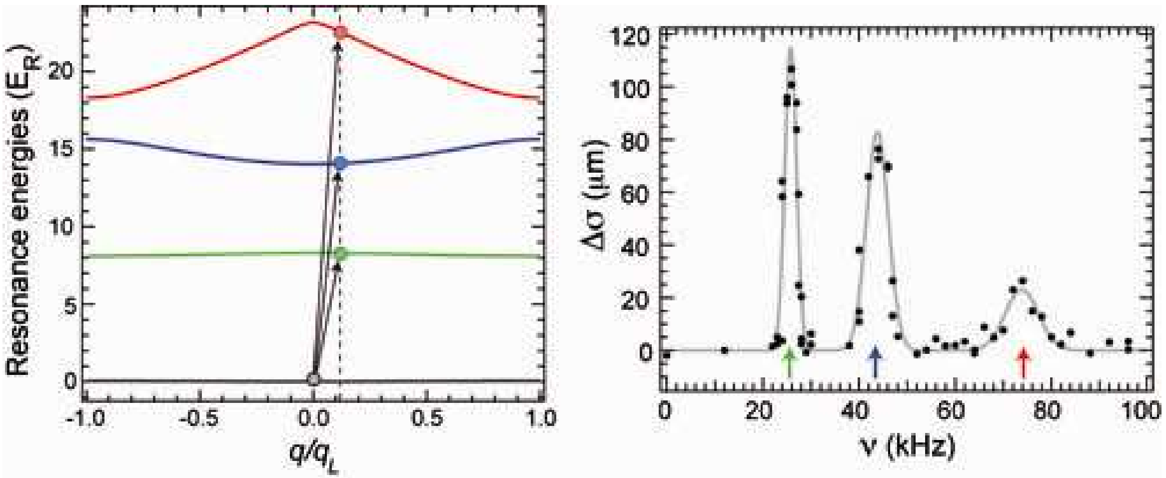


Center-of-mass velocity *vs* time.



Excitations of a BEC in a lattice

The band structure of the Bogoliubov excitations can also be measured by means of light (**Bragg**) scattering as in recent experiment in Florence (Fabbri et al., 2009; Clement et al., 2009).



Let us come back to

Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

Large *s*: **tight-binding limit**

$$\varepsilon(P)=2\delta_J\sin^2(Pd/2\hbar)$$

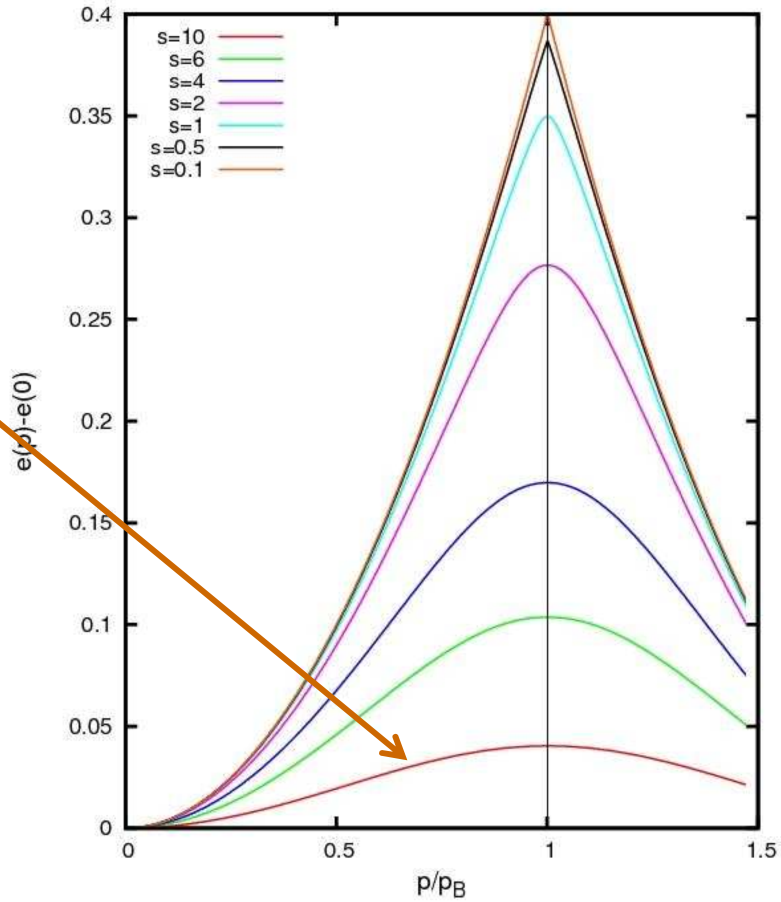
$$\delta_J=\frac{\hbar^2}{m^*d^2}$$

tunnelling energy

In this regime the flow is dominated by macroscopic tunnelling between lattice sites.

average density

Lowest Bloch band ($gn=0.4E_R$)



Let us come back to

Bloch waves and bands

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}-i\frac{P}{\hbar}\right)^2\varphi_P(z)+\left[g\left|\varphi_P(z)\right|^2+V_{opt}(z)\right]\varphi_P(z)=\mu(P)\varphi_P(z)$$

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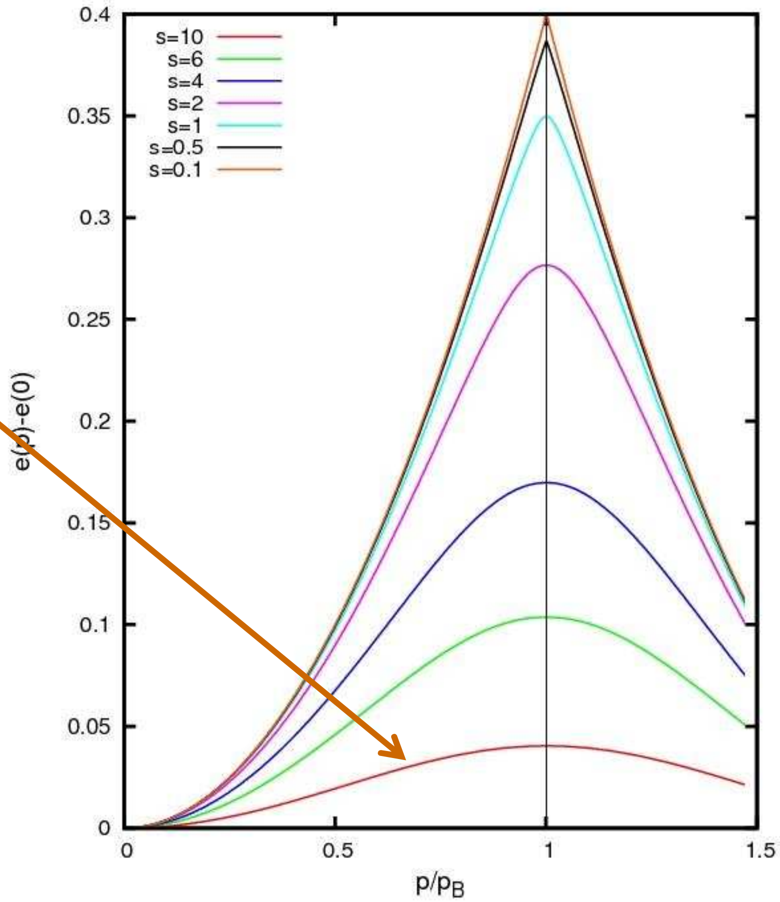
tunnelling energy

In this regime the flow is dominated by macroscopic tunnelling between lattice sites.

We want to understand the connection with the physics of Josephson effect

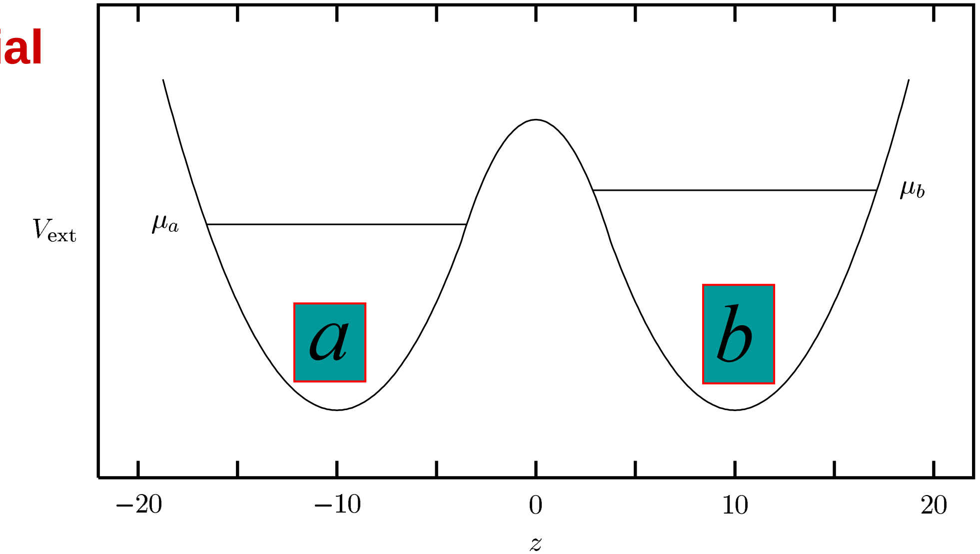
average density

Lowest Bloch band ($gn=0.4E_R$)



Josephson oscillations

BEC in a double well potential



Nonstationary solutions of GP equation in the form

$$\Psi(z,t) = \Psi_a(z; N_a)e^{iS_a} + \Psi_b(z; N_b)e^{iS_b}$$

where $N = N_a + N_b$ is constant.

Assumption: small overlap between two BECs under the barrier.

Important results: atomic current associated with phase difference**!!**

$$I = \frac{\partial N_a}{\partial t} = -\frac{\partial N_b}{\partial t}$$

$$\phi = S_a - S_b$$

at *z=0*

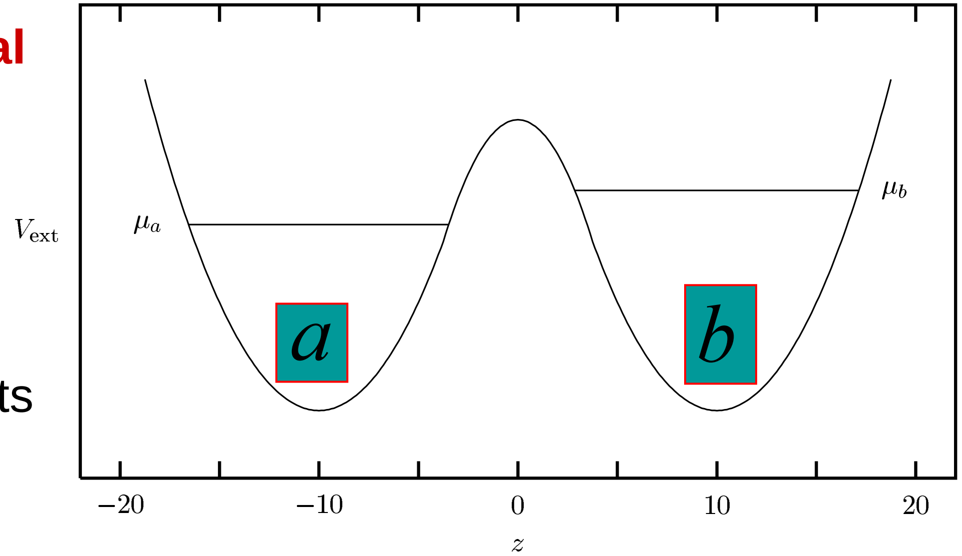
$$I = -I_j \sin\phi$$

Josephson, 1962

with $$I_j = \frac{\hbar}{m}\left[\Psi_a \frac{\partial \Psi_b}{\partial z} - \Psi_b \frac{\partial \Psi_a}{\partial z}\right]_{z=0}$$

Josephson oscillations

BEC in a double well potential



Now recall the phase equation

$$\hbar\frac{\partial}{\partial t}S = -\left(\frac{1}{2}mv_S^2 + V_{ext} + \mu\right)$$

and neglect v^2 (small currents). One gets

$$\frac{\partial\phi}{\partial t} = -\frac{1}{\hbar}(\mu_a - \mu_b)$$

Then define

$$k = (N_a - N_b)/2$$

and expand μ with respect to k. One gets $$\frac{\partial\phi}{\partial t} = -\frac{E_C}{\hbar}k$$ with $$E_C = 2\frac{d\mu_a}{dN_a}$$

Moreover, this equation $I = -I_j \sin\phi$ becomes $$\frac{dk}{dt} = -I_j \sin\phi$$

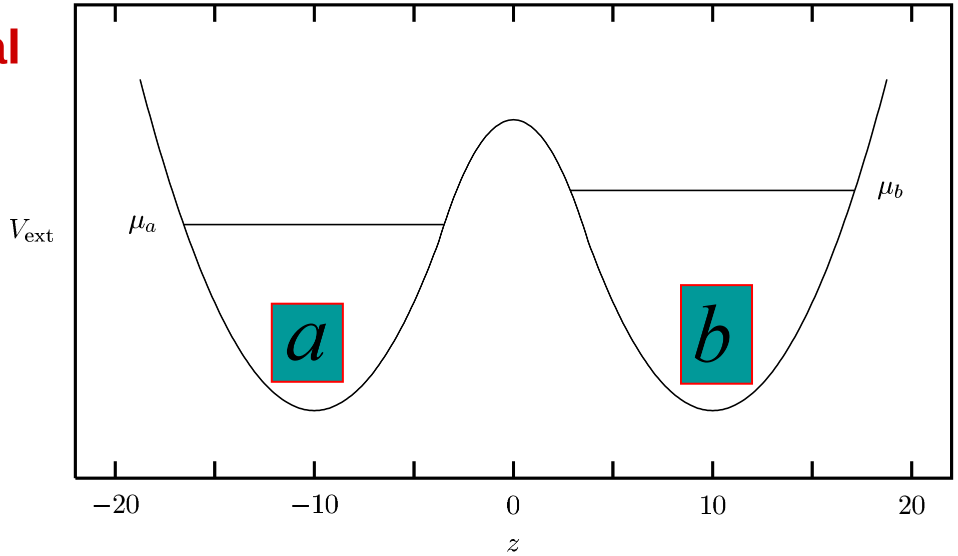
Josephson oscillations

BEC in a double well potential

$$\frac{\partial \phi}{\partial t} = -\frac{E_C}{\hbar} k$$

$$\frac{dk}{dt} = -\frac{E_J}{\hbar} \sin\phi$$

$$\phi = S_a - S_b$$
$$k = (N_a - N_b)/2$$
$$E_C = 2\frac{d\mu_a}{dN_a}$$
$$E_J = \hbar I_J$$



These two equations are valid for small overlap, small current, large N_a and N_b, small $(N_a\text{-}N_b)$.

They can be rewritten in Hamiltonian form:

Josephson Hamiltonian

$$H_J = \frac{1}{2} E_C k^2 - E_J \cos\phi$$

⇒

$$\frac{\partial \hbar k}{\partial t} = -\frac{\partial H_J}{\partial \phi}$$
$$\frac{\partial \phi}{\partial t} = \frac{\partial H_J}{\partial (\hbar k)}$$

Small oscillations: $\hbar\omega = \sqrt{E_C E_J}$

Note: $\hbar k$ and ϕ play the role of canonically conjugate variables**!**

Josephson oscillations

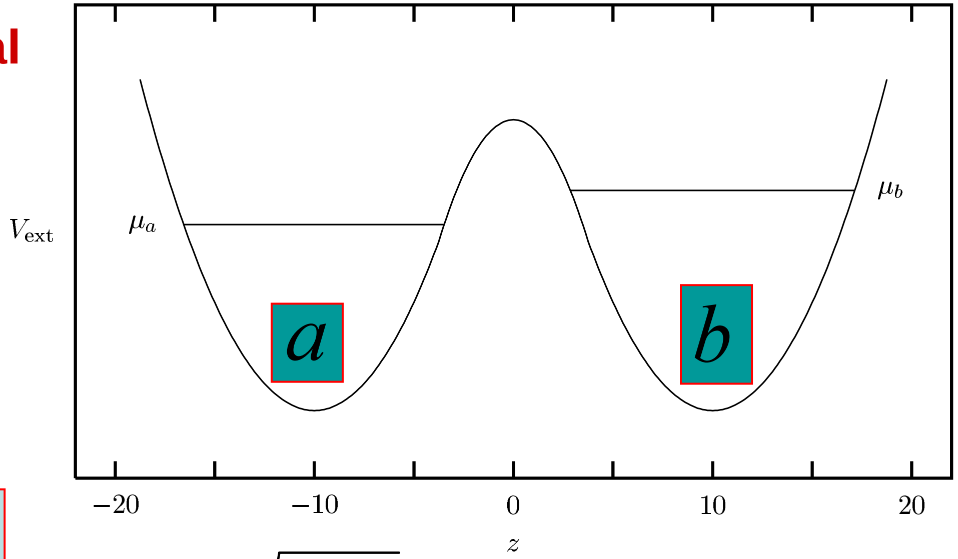
BEC in a double well potential

A more accurate form of E_J, including a correct k dependence (coming from the dependence of $\Psi_{a(b)}$ on $N_{a(b)}$):

$$E_J = (\delta_J / 2)\sqrt{N^2 - 4k^2}$$

with

$$\delta_J = \frac{\hbar^2}{m}\left[\psi_a \frac{\partial \psi_b}{\partial z} - \psi_b \frac{\partial \psi_a}{\partial z}\right]_{z=0}$$



$$\Psi_{a(b)} = \sqrt{N_{a(b)}}\,\psi_{a(b)}$$

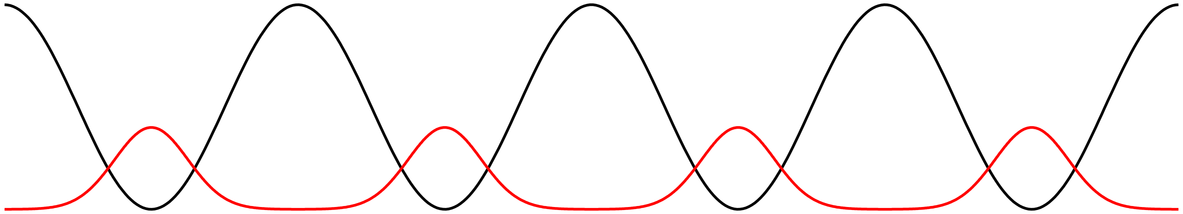
Josephson Hamiltonian:

$$H_J = \frac{E_C}{2}k^2 - \frac{\delta_J}{2}\sqrt{N^2 - 4k^2}\cos\phi$$

$$\frac{\partial \hbar k}{\partial t} = -\frac{\partial H_J}{\partial \phi}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial H_J}{\partial (\hbar k)}$$

Josephson oscillations in a lattice



A generalization of the previous calculations gives the Josephson Hamiltonian:

$$H_J = -\frac{E_C}{4}\sum_l (N_l^{'})^2 - \delta_J \sum_l \sqrt{(N_0 + N_{l+1}^{'})(N_0 + N_l^{'})}\cos(S_{l+1} - S_l)$$

where

$$N_l^{'} = N_l - N_0$$

N_l: number of atoms in site ℓ

N_0: average (equilibrium) number of atoms in site ℓ

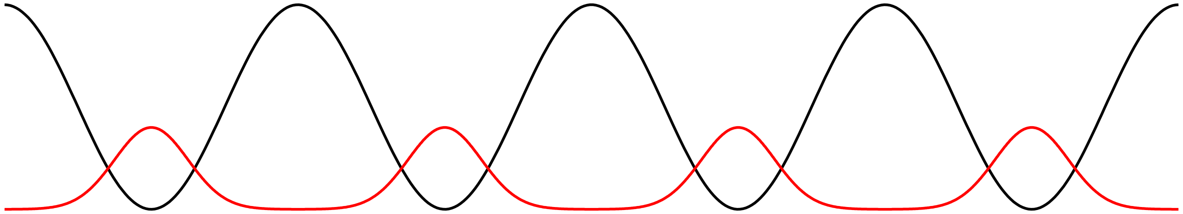
$$E_C = 2d\mu_l / dN_l$$

on-site energy parameter (or charging energy)

$$\delta_J = \frac{\hbar^2}{m}\left[\psi_l \frac{\partial \psi_{l+1}}{\partial z} - \psi_{l+1}\frac{\partial \psi_l}{\partial z}\right]$$

Tunnelling energy parameter
(approximation: only tunnelling between adjacent sites)

Josephson oscillations in a lattice



$$H_J = -\frac{E_C}{4}\sum_l (N_l^{'})^2 - \delta_J \sum_l \sqrt{(N_0 + N_{l+1}^{'})(N_0 + N_l^{'})}\cos(S_{l+1} - S_l)$$

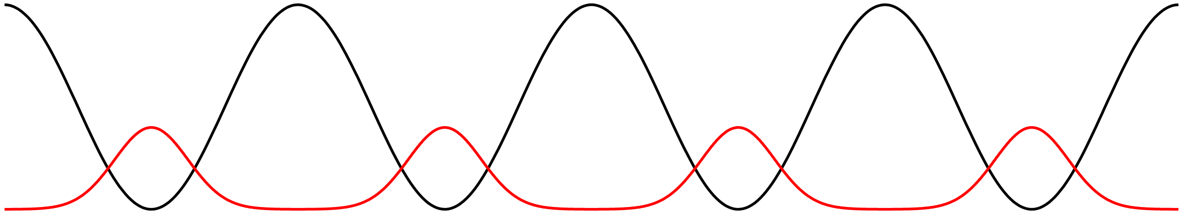
Equilibrium: $N_l^{'} = 0\,,\quad S_l = const$

Small oscillations around equilibrium:

$$\frac{\partial S_l}{\partial t} = -\frac{E_C}{2\hbar} N_l^{'} + \frac{\delta_J}{4N_0}(N_{l+1}^{'} - 2N_l^{'} + N_{l-1}^{'})$$

$$\hbar \frac{\partial N_l^{'}}{\partial t} = -N_0 \delta_J (S_{l+1} - 2S_l + S_{l-1})$$

Josephson oscillations in a lattice



$$\frac{\partial S_l}{\partial t} = -\frac{E_C}{2\hbar} N_l' + \frac{\delta_J}{4N_0}(N_{l+1}' - 2N_l' + N_{l-1}')$$

$$\hbar \frac{\partial N_l'}{\partial t} = -N_0 \delta_J (S_{l+1} - 2S_l + S_{l-1})$$

quasi-momentum of the excitation

energy of the excitation

By looking for periodic solutions $S_l(t),\ N_l'(t) \propto \exp[i(lpd - \varepsilon_{exc}(p)t)/\hbar]$

one finds the dispersion relation in tight binding limit (Javanainen 1999)

$$\varepsilon_{exc}^2(p) = N_0 E_C \varepsilon_0(p) + \varepsilon_0^2(p)$$

with $\varepsilon_0(p) = 2\delta_J \sin^2(pd/2\hbar)$

This is the spectrum of excitations. It includes the free particle limit ($E_C=0$) and the phononic limit (long wavelength), in a lattice.

Josephson oscillations in a lattice

Large *s*: **tight-binding limit**

Energy per particle of BEC moving in the lattice with quasi-momentum *P*

$$\varepsilon(P) = 2\delta_J \sin^2(Pd/2\hbar)$$

Same δ_j

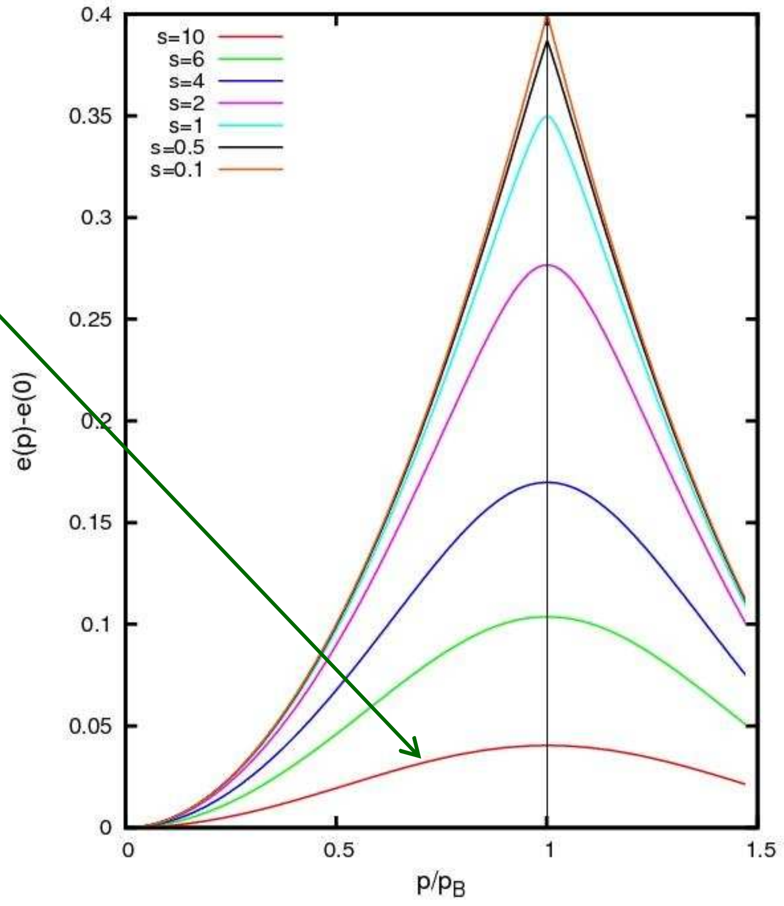
$$\delta_J = \frac{\hbar^2}{m^* d^2}$$

Energy of single-particle excitations with quasi-momentum *p* in a BEC at rest:

$$\varepsilon_0(p) = 2\delta_J \sin^2(pd/2\hbar)$$

In this regime the flow is dominated by (Josephson) tunnelling between lattice sites.

Lowest Bloch band ($gn=0.4E_R$)

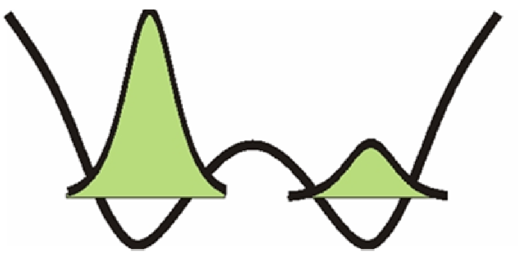


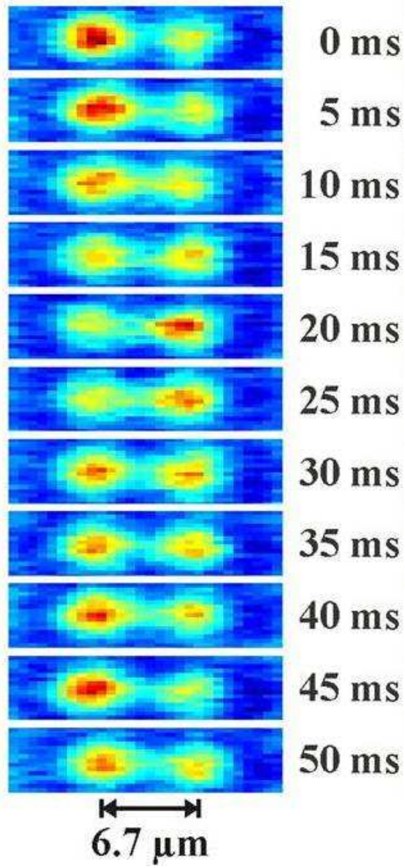
Josephson oscillations in a lattice

The physics of the Josephson effect is the subject of a huge field of investigations:

- in superconducting devices (Josephson junctions)
- in superfluids (3He and He4, ultracold gases)

Recent experiments by M. Oberthaler et al. with BECs in a double well





Josephson oscillations in a lattice

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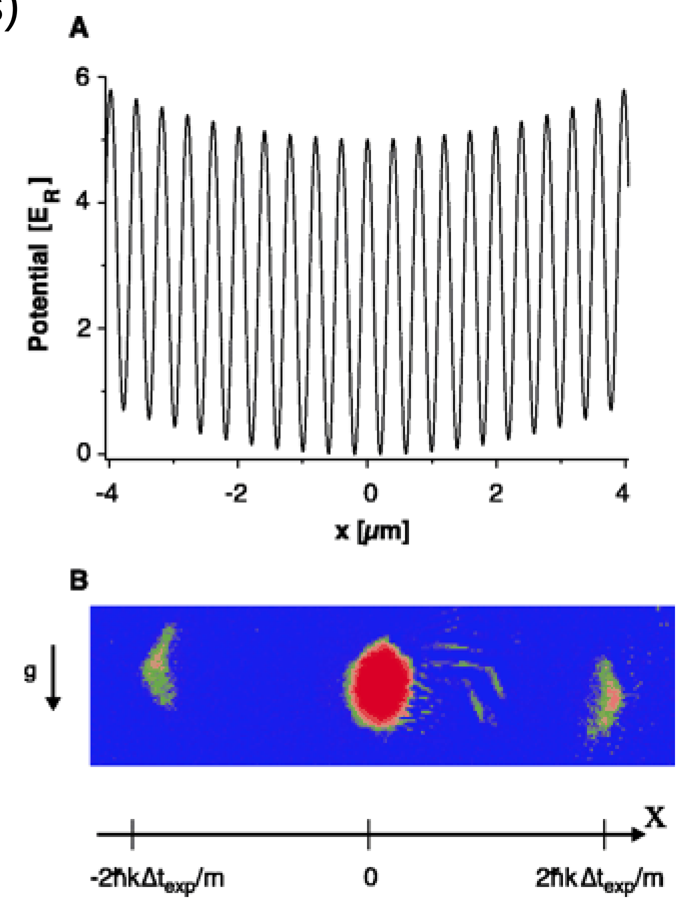


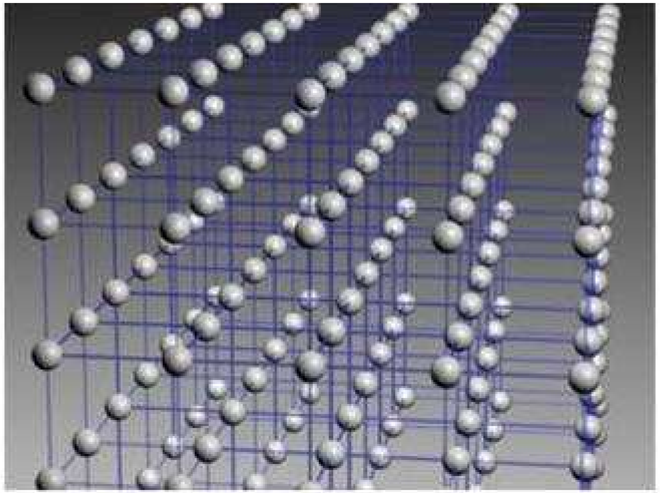
Experiments at LENS-Florence with a BEC in an optical lattice in the regime of weakly coupled condensates (Josephson Junction Arrays with Bose-Einstein Condensates, Science, 2001)

From superfluid to Mott insulator

What we have seen so far is valid if the number of particles in each lattice site is large (i.e., well defined BEC density and phase, GP theory works, etc.).

If the **number** of particles **per site** is of order of **unity** the formalism of **Josephson** Hamiltonian is **no longer adequate**.

This is usually the case in 3D optical lattices.



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One has to start again from the full quantum Hamiltonian

$$\hat{H} = \int d\mathbf{r}\,\hat{\Psi}^{+}(\mathbf{r})\left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r})\right]\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\iint d\mathbf{r}d\mathbf{r}'\,\hat{\Psi}^{+}(\mathbf{r})\hat{\Psi}^{+}(\mathbf{r}')V(|\mathbf{r}-\mathbf{r}'|)\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})$$

use the potential $V_{eff} = g\delta(r-r')$

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$$\hat{H} = \int d\mathbf{r}\, \hat{\Psi}^{+}(\mathbf{r}) \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}) + \frac{g}{2} \int d\mathbf{r}\, \hat{\Psi}^{+}(\mathbf{r}) \hat{\Psi}^{+}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r})$$

Then remember that we are in lattice and write the field operators using a basis of single site operators:

$$\hat{\Psi}^{+} = \sum_k \varphi_k \hat{a}_k^{+}$$

This creates a particle in the *k*-site

By ignoring all interaction terms except those involving nearest neighbors, one obtains

Pairs of nearest neighbors

Bose-Hubbard Hamiltonian

$$\hat{H} = \frac{E_C}{4} \sum_k \hat{n}_k (\hat{n}_k - 1) - \frac{\delta_J}{2} \sum_{<k,l>} (\hat{a}_k^{+} \hat{a}_l + \hat{a}_l^{+} \hat{a}_k)$$

$$\hat{n}_k^{+} = \hat{a}_k^{+} \hat{a}_k$$

On-site interaction

$$E_C = 2g \int d\mathbf{r}\, |\varphi_k|^4$$

Tunnelling parameter

$$\delta_J = -2 \int d\mathbf{r}\, \varphi_k \left[-(\hbar^2 / 2m) \nabla^2 + V_{ext} \right] \varphi_l$$

From superfluid to Mott insulator

Bose-Hubbard Hamiltonian

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- The phase diagram of B-H Hamiltonian exhibits a **superfluid- Mott insulator transition** for **integer** values of the average **occupation number** per site.

- Superfluid phase corresponds to non vanishing of average

$$\langle\hat{\Psi}\rangle = \left\langle\sum_k \varphi_k \hat{a}_k\right\rangle \neq 0$$

Order parameter

- For occupation number =1 many-body theory theory predicts quantum phase transition at critical value (Fisher et al. 1989)

$$E_C/\delta_J = 34.8$$

- For larger values of E_C/δ_J **insulator** phase (**no long range order**).

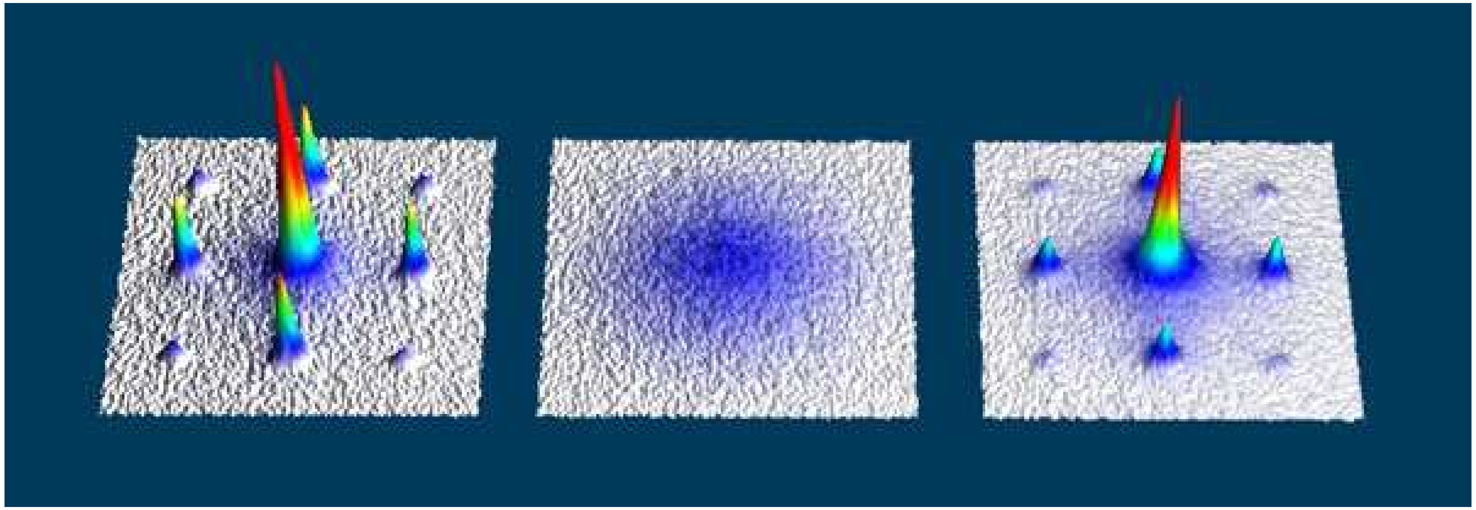
- For smaller values **superfluid** phase (**long range order**).

From superfluid to Mott insulator

- Extension of theory to harmonic trapping: Jacksch et al. ,1998.

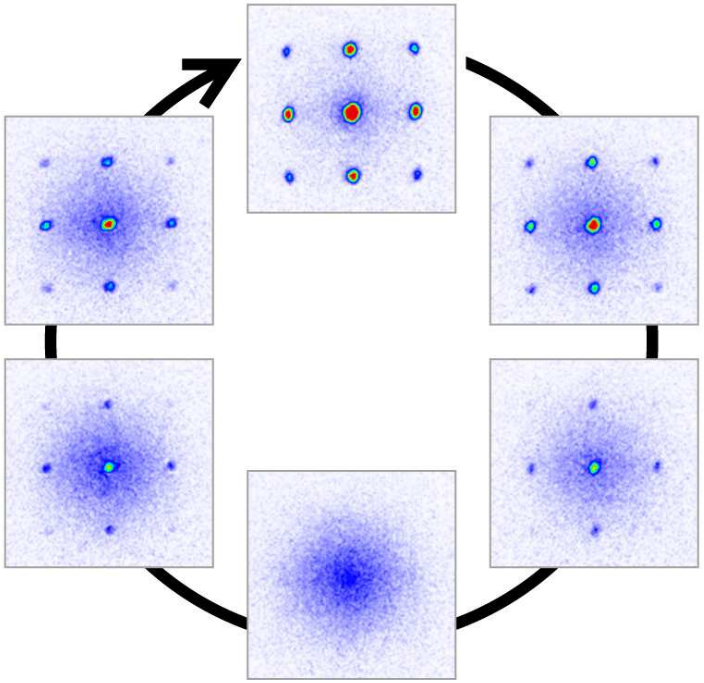
- In Bose gases, the superfluid phase can be tested by measuring interference patterns in expanding condensates. Interference is the result of the occurrence of an order parameter and reflects its coherent behaviour in **momentum space.**

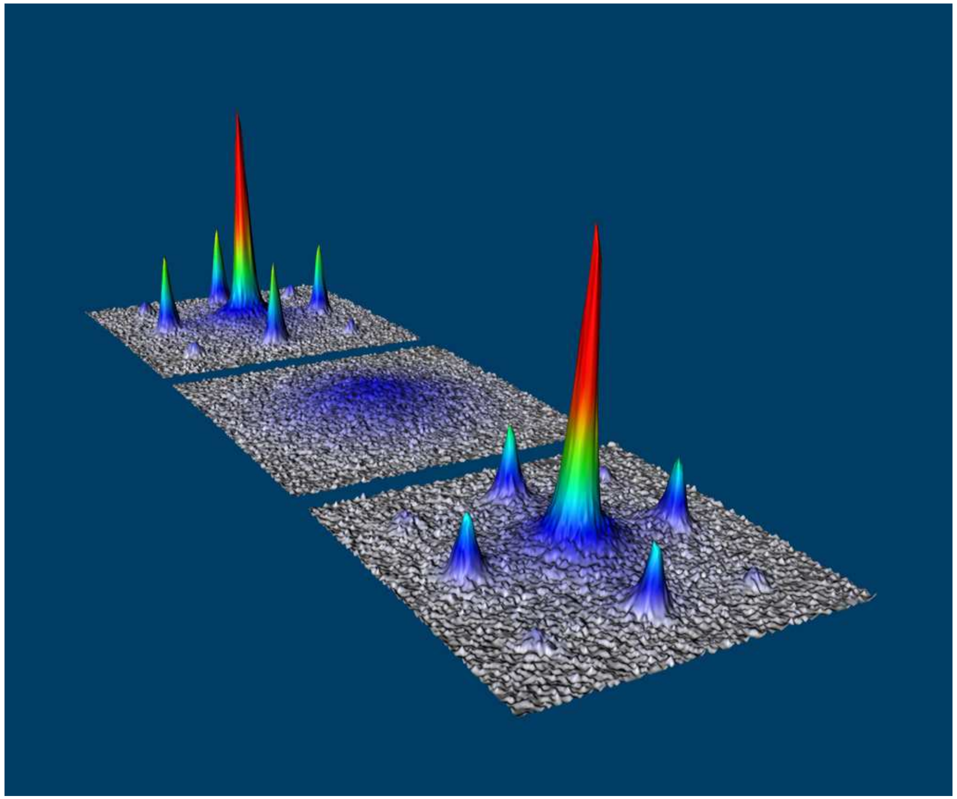
- **Disappearence of fringes** at large lattice intensities reveals the occurrence of the **transition to the Mott insulating phase**.



(I.Bloch et al. Nature 2002)

superfluid Mott superfluid





What next:

Ultracold Fermions and Bogoliubov - de Gennes theory