Stationary
GP equation

Equation for the order parameter

$$i\hbar\frac{\partial}{\partial t}\Psi_0(\mathbf{r},t)=\left[-\frac{\hbar^2\nabla^2}{2m}+V_{ext}(\mathbf{r})+g\left|\Psi_0(\mathbf{r},t)\right|^2\right]\Psi_0(\mathbf{r},t)$$

Gross-Pitaevskii equation

From lecture #1

Stationary GP

By inserting this

$$\Psi_0(\mathbf{r},t) = e^{-i\mu t/\hbar}\,\Psi_0(\mathbf{r})$$

into the GP equation

$$i\hbar\frac{\partial}{\partial t}\Psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left|\Psi_0(\mathbf{r},t)\right|^2\right]\Psi_0(\mathbf{r},t)$$

one finds the stationary GP equation:

$$\boxed{\left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left|\Psi_0(\mathbf{r})\right|^2\right]\Psi_0(\mathbf{r}) = \mu\Psi_0(\mathbf{r})}$$

It gives the **ground state** of the condensate and all possible **stationary** states (vortices, solitons, etc.)

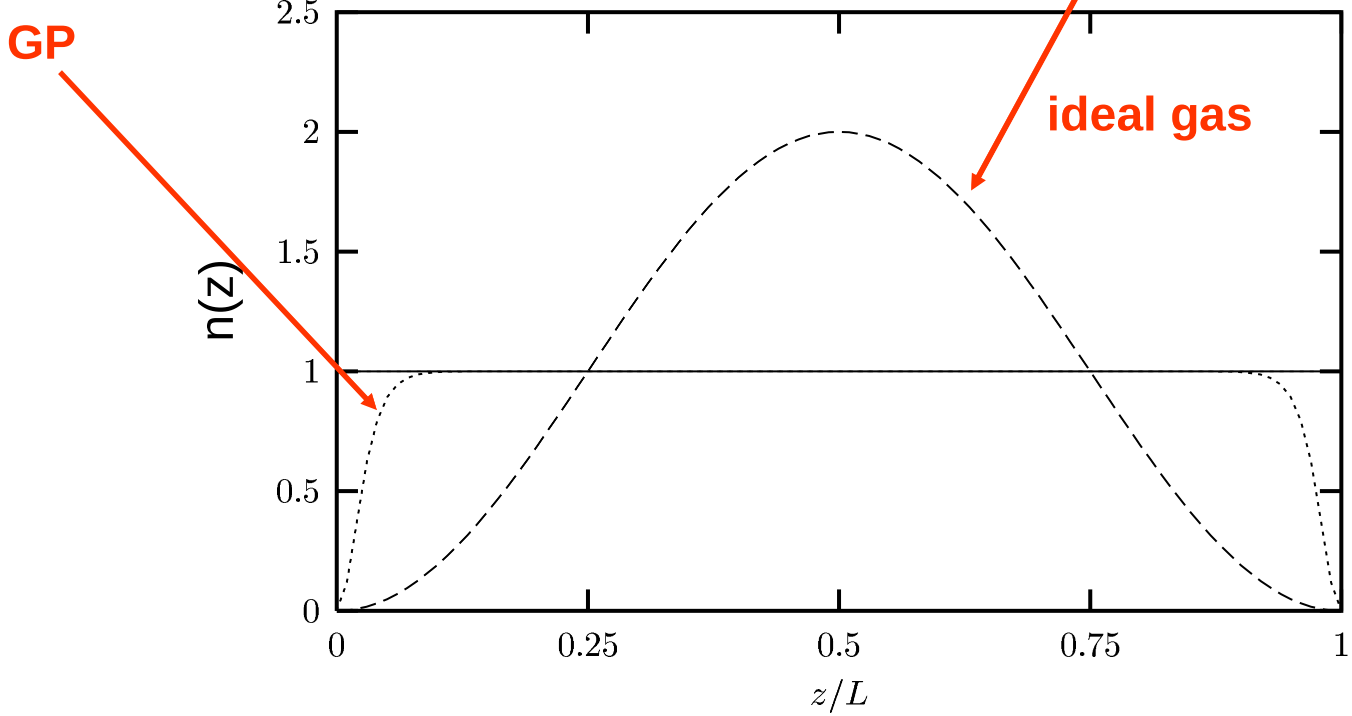
Stationary GP: BEC in a box

Example: 1D box of size *L* and hard walls.

Solution of Schrödinger equation for free particles: $\boxed{\Psi_0 = \sqrt{2\bar{n}}\sin(\pi z / L)}$

average density

GP equation with $a > 0$



Stationary GP: BEC in a box

In order to stress the role of interaction in GP, let us rescale the units:

$$\Psi_0 \to \frac{1}{\sqrt{\bar{n}}}\Psi_0 \quad , \quad z \to \frac{z}{\xi}$$

where

$$\boxed{\xi = \sqrt{\frac{\hbar^2}{2mg\bar{n}}} = \sqrt{\frac{1}{8\pi a\bar{n}}}}$$

healing length

The GP equation becomes:

$$-\frac{d^2}{dz^2}\Psi_0(z) + \Psi_0^{\,3}(z) = \Psi_0(z)$$

If $L >> \xi$ one can use the boundary conditions: $\Psi_0(0) = 0, \Psi_0(\infty) = 1$

and the solution is:

$$\boxed{\Psi_0(z) = \sqrt{\bar{n}}\tanh\frac{z}{\sqrt{2}\xi}}$$

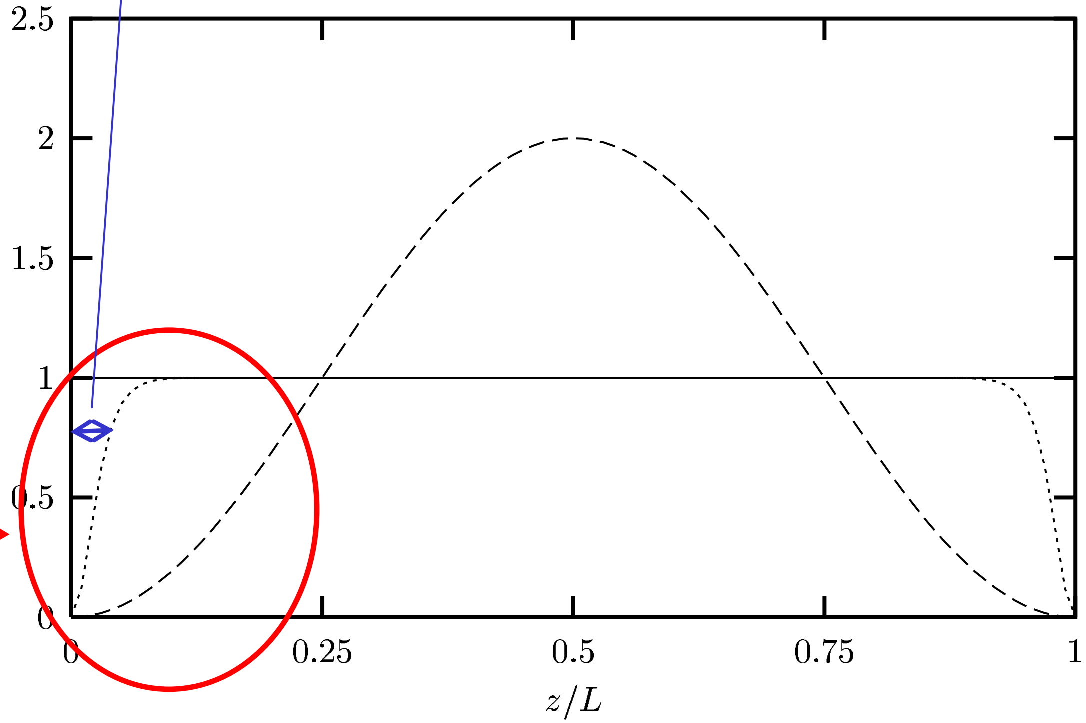
Stationary GP: BEC in a box

$$\xi = \sqrt{\frac{\hbar^2}{2mg\bar{n}}} = \sqrt{\frac{1}{8\pi a\bar{n}}}$$

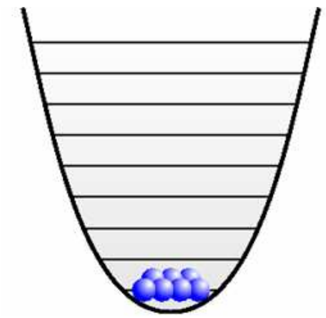
healing length

crucial parameter characterizing the interaction

$$\Psi_0(z) = \sqrt{\bar{n}} \tanh \frac{z}{\sqrt{2}\xi}$$



If $L >> \xi$ GP predictions differ significantly from those of an ideal gas !



Stationary GP: harmonic trap

$$V_{ext} = \frac{1}{2} m \omega_{ho}^2 r^2$$

Noninteracting ground state: $\Psi_0(r) \propto \exp(-r^2 / a_{ho}^2)$

$$a_{ho} = \sqrt{\hbar / m\omega_{ho}}$$

depends on ħ

Role of interactions:

Using $\boxed{a_{ho}}$ and $\boxed{\hbar\omega_{ho}}$ as units of lengths and energy, and $\boxed{\tilde{\Psi} = N^{-1/2} a_{ho}^{-3/2} \Psi_0}$

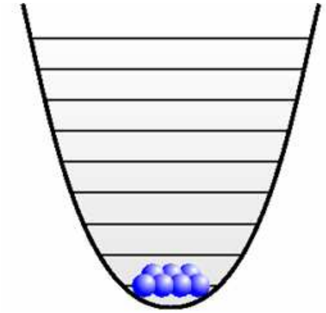
GP equation becomes

$$[-\tilde{\nabla}^2 + \tilde{r}^2 + 8\pi (Na / a_{ho}) \tilde{\Psi}^2(\tilde{r})] \tilde{\Psi}(\tilde{r}) = 2\tilde{\mu} \tilde{\Psi}(\tilde{r})$$

Thomas-Fermi parameter

If $Na / a_{ho} << 1$ **Noninteracting** ground state

If $Na / a_{ho} >> 1$ **Thomas-Fermi** limit (a>0)



Stationary GP: harmonic trap

If $Na / a_{ho} >> 1$

$$\left[\cancel{-\frac{\hbar^2\nabla^2}{2m}} + V_{ext}(\mathbf{r}) + g|\Psi_0(\mathbf{r})|^2\right]\Psi_0(\mathbf{r}) = \mu\Psi_0(\mathbf{r})$$

and thus

$$\boxed{|\Psi_0(\mathbf{r})|^2 = n(\mathbf{r}) = \frac{1}{g}[\mu - V_{ext}(\mathbf{r})]}$$

Thomas-Fermi density profile

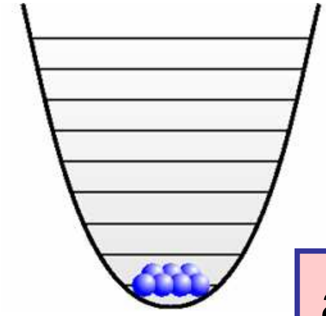
In an isotropic harmonic potential the density is an inverted parabola with radius $\boxed{R = a_{ho}(15Na/a_{ho})^{1/5}}$

The chemical potential is fixed by the normalization to N:

$$\boxed{\mu = gn(0) = (1/2)\hbar\omega_{ho}(15Na/a_{ho})^{2/5}}$$

The Thomas-Fermi $Na / a_{ho} >> 1$ limit implies:

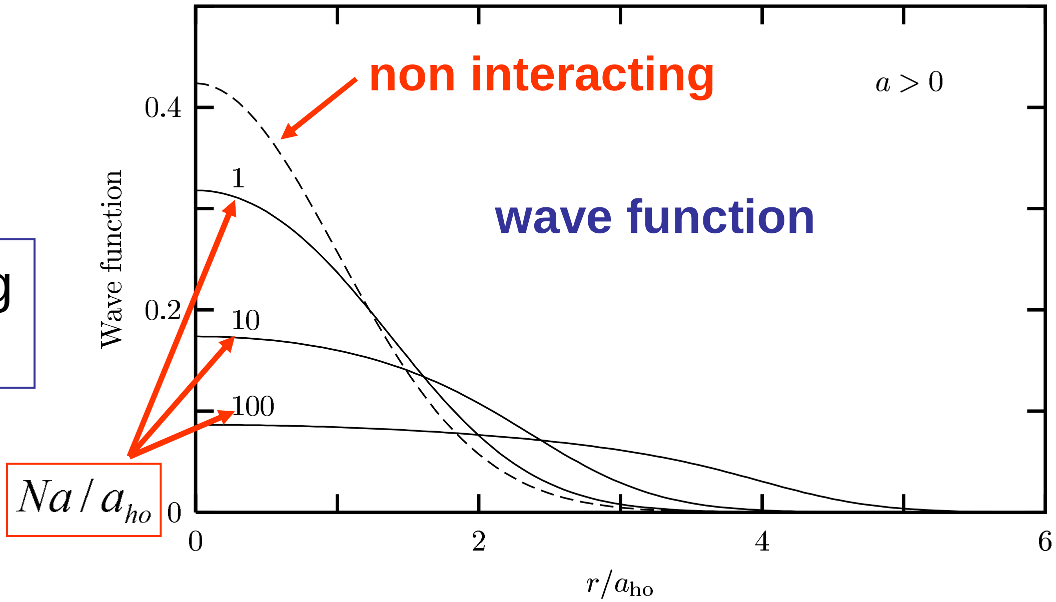
$$\mu >> \hbar\omega_{ho} \quad , \quad R >> a_{ho} \quad , \quad R >> \xi$$



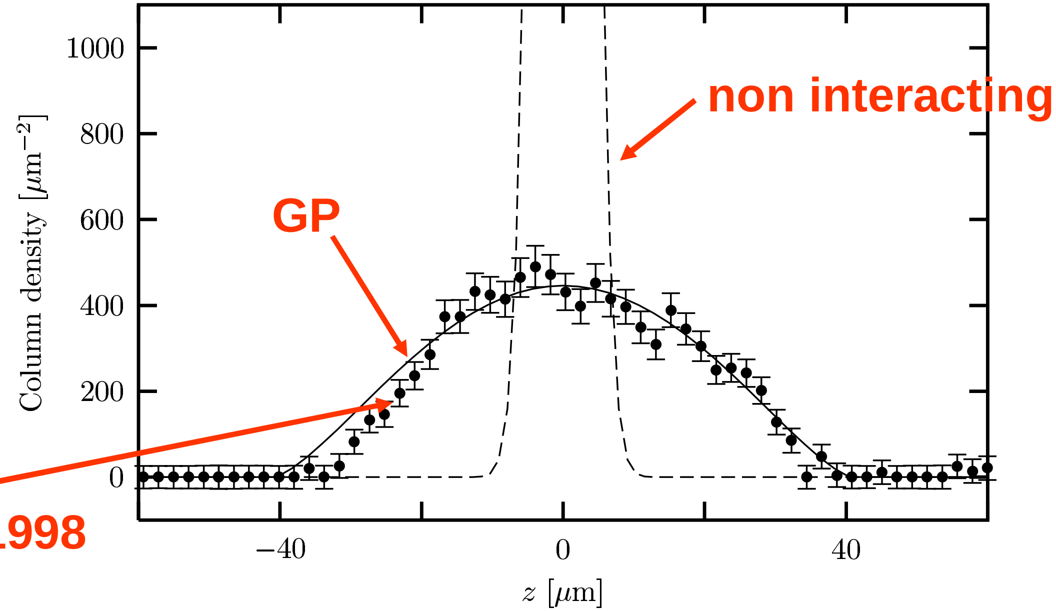
Stationary GP: harmonic trap

a >0

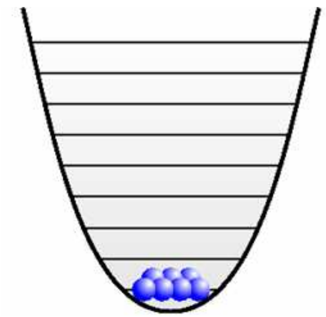
from noninteracting
to Thomas-Fermi:



Large effects due to **interaction** at equilibrium; **good agreement** with **experiments**



exp: Hau et al, 1998



Stationary GP: harmonic trap

Note: Thomas-Fermi regime is compatible with diluteness condition

Gas parameter in the center of the trap → $na^3 = \mu a^3 / g = 0.1(N^{1/6} a / a_{ho})^{12/5}$

Thomas-Fermi

$$Na / a_{ho} >> 1$$

Diluteness

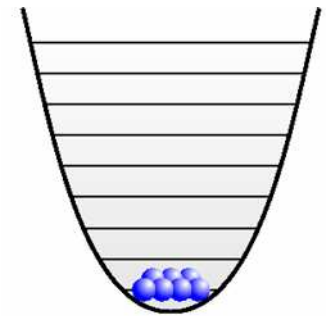
$$N^{1/6} a / a_{ho} << 1$$

example: $a / a_{ho} = 10^{-3}, \; N = 10^6$

$$Na / a_{ho} = 10^3$$

$$N^{1/6} a / a_{ho} = 10^{-2}$$

Gross-Pitaevskii theory is not perturbative even if the gas is dilute (role of BEC)!



Stationary GP: harmonic trap

a < 0

For **attractive** force TF limit is not available.
For **large N** the system is **unstable** (**negative compressibility**). **Kinetic energy term** term is crucial to ensure metastable solution at finite N.

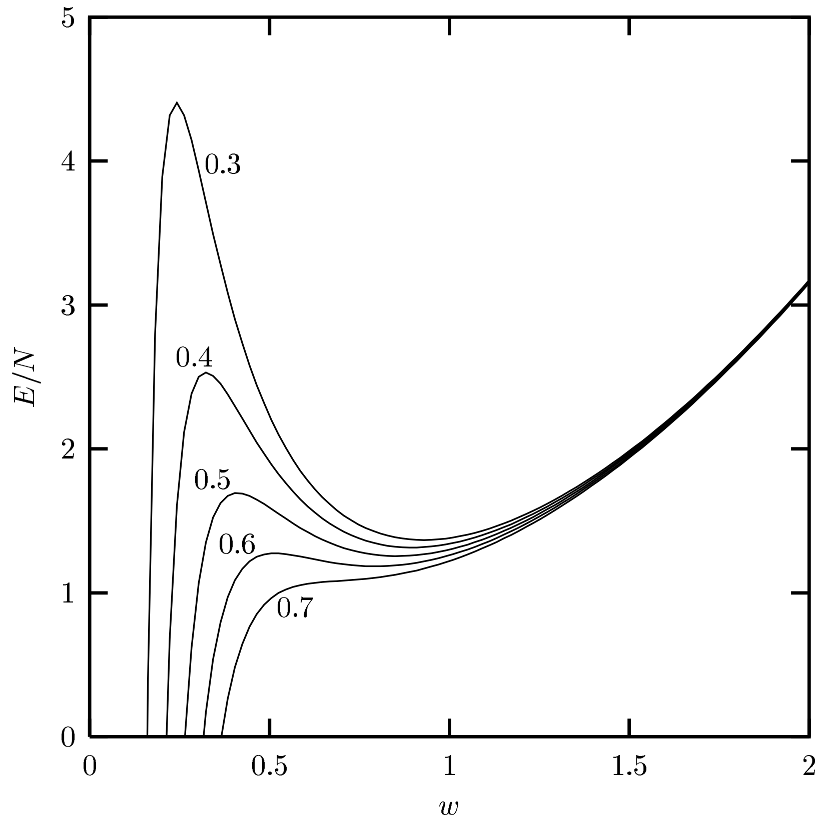
No stationary solution in a spherical trap if

$$\frac{N|a|}{a_{ho}} < 0.58$$

Physical insight provided by variational approach based on Gaussian function:

$$\psi(r) = \frac{N^{1/2}}{w^{3/2} a_{ho}^{3/2} \pi^{3/2}} e^{-r^2/2w^2 a_{ho}^2}$$

width of gaussian:
variational parameter



First experiments on collapse in ^{85}Rb (JILA, 2001)

Time-dependent GP equation

Time-dependent Gross-Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t}\Psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left|\Psi_0(\mathbf{r},t)\right|^2 \right] \Psi_0(\mathbf{r},t)$$

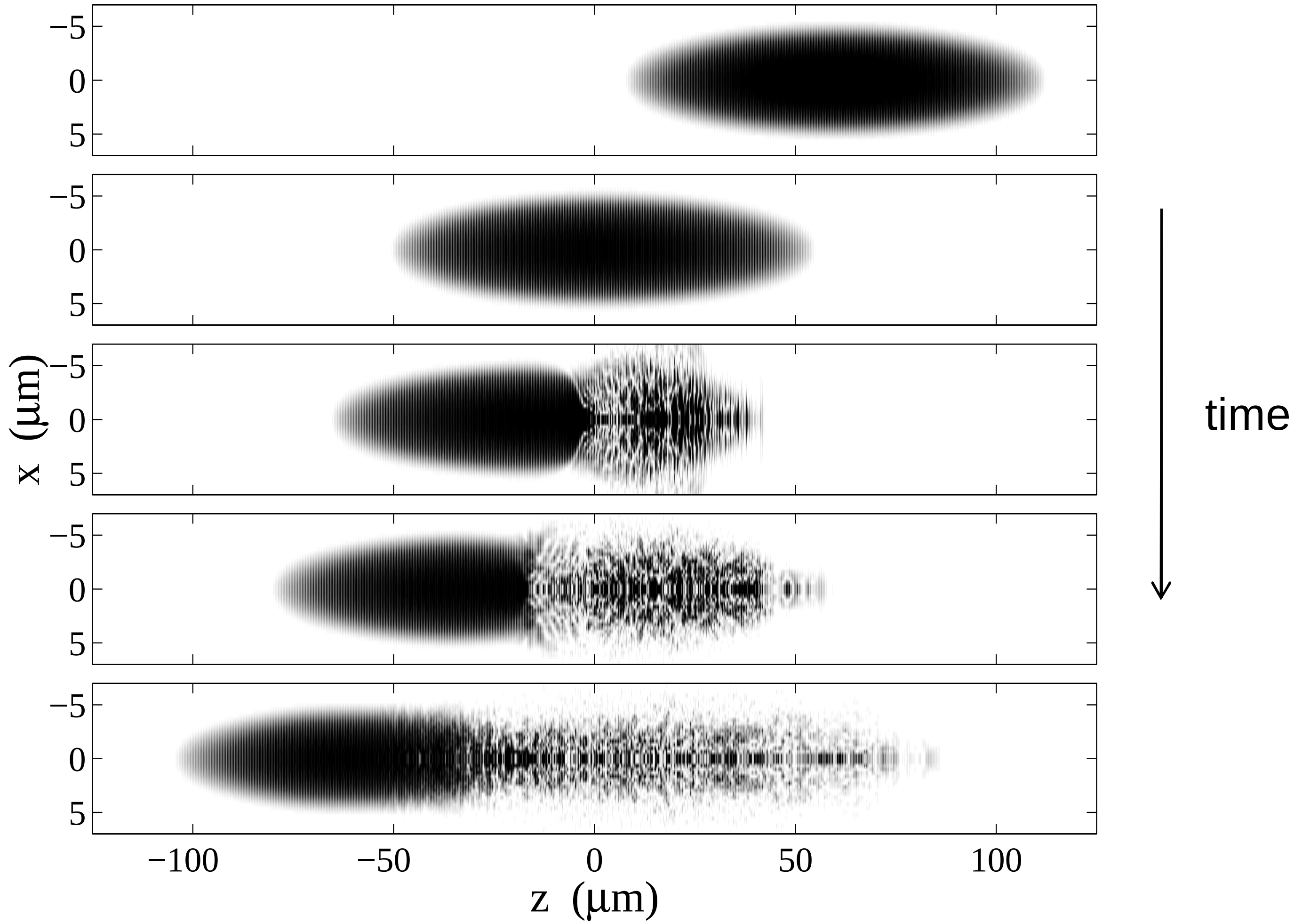
This equation can be

- ✓ Numerically solved (GP simulations)
- ✓ Linearized for small oscillations (Bogoliubov equations)
- ✓ Rewritten in terms of density and velocity (T=0 hydrodynamics)

Time-dependent Gross-Pitaevskii equation

Numerical integration.

Example: a BEC oscillating in a trap + optical lattice. Onset of instabilities.



Time-dependent Gross-Pitaevskii equation

Linearization for small oscillations

Ansatz: $$\Psi_0(\mathbf{r},t) = e^{-i\mu t}[\,\Psi_0(\mathbf{r}) + u_j(\mathbf{r})e^{-i\omega_j t} + v_j^*(\mathbf{r})e^{i\omega_j t}\,]$$

Zero-order in *u* and *v*: $$\left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left|\Psi_0(\mathbf{r})\right|^2\right]\Psi_0(\mathbf{r}) = \mu\Psi_0(\mathbf{r})$$

First-order in *u* and *v*:

$$\hbar\omega_j u_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)u_j + g\Psi_0^2 v_j$$

$$-\hbar\omega_j v_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)v_j + g\Psi_0^{*2} u_j$$

Bogoliubov equations !

Bogoliubov equations

$$\Psi_0(\mathbf{r},t) = e^{-i\mu t}[\,\Psi_0(\mathbf{r}) + u_j(\mathbf{r})e^{-i\omega_j t} + v_j^*(\mathbf{r})e^{i\omega_j t}\,]$$

Bogoliubov eqs:

$$\hbar\omega_j u_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)u_j + g\Psi_0^2 v_j$$

$$-\hbar\omega_j v_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)v_j + g\Psi_0^{*2} u_j$$

u and *v* are Bogoliubov quasiparticle amplitudes.

$\hbar\omega$ are quasiparticle energies.

n_0 is the ground state density: $n_0(\mathbf{r}) = |\Psi_0(\mathbf{r})|^2$

Bogoliubov equations

$$\Psi_0(\mathbf{r},t) = e^{-i\mu t}[\Psi_0(\mathbf{r}) + u_j(\mathbf{r})e^{-i\omega_j t} + v_j^*(\mathbf{r})e^{i\omega_j t}]$$

Bogoliubov eqs:

$$\hbar\omega_j u_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)u_j + g\Psi_0^2 v_j$$

$$-\hbar\omega_j v_j = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2gn_0\right)v_j + g\Psi_0^{*2} u_j$$

Note: the **same equations** can be also derived diagonalizing **quantum** Hamiltonian using **Bogoliubov transformations**.

interacting particles → noninteracting quasiparticles

Bogoliubov equations

Properties of *u* and *v:*

$$(\omega_i - \omega_i^*) \int d\mathbf{r} \left(|u_i|^2 - |v_i|^2\right) = 0 \quad \Rightarrow \omega_i$$ are real, unless $\int d\mathbf{r}\, |u_i|^2 = \int d\mathbf{r}\, |v_i|^2$

occurrence of complex solutions $\Longrightarrow$ **dynamic instability**

$$\int d\mathbf{r}\, (u_i u_j^* - v_i v_j^*) = \delta_{ij}$$ orthogonality and normalization

For each solution u_i, v_i, ω_i there exists another solution with $v_i^*, u_i^*, -\omega_i$
(the two solutions describe the same physical oscillation)

If $\Psi_0(\mathbf{r},t) = e^{-i\mu t}(\Psi_0 + u_j e^{-i\omega_j t} + v_j^* e^{i\omega_j t})$, with u_j, v_j, ω_j solution of Bogoliubov eqs., then the energy change with respect to equilibrium is:

$$\delta E = \hbar\omega_j \int d\mathbf{r} \left(|u_j|^2 - |v_j|^2\right)$$

Condition of **energetic stability** $\delta E > 0 \quad \Rightarrow \quad \omega_j \int d\mathbf{r}\, (|u_j|^2 - |v_j|^2) > 0$

Bogoliubov equations

Solutions of Bogoliubov equations in a uniform gas: $u, v \propto e^{i\mathbf{q}\cdot\mathbf{r}}$

Bogoliubov dispersion law

$$\boxed{\omega^2 = \hbar^2 \left(\frac{q^2}{2m}\right)^2 + q^2 c^2}$$

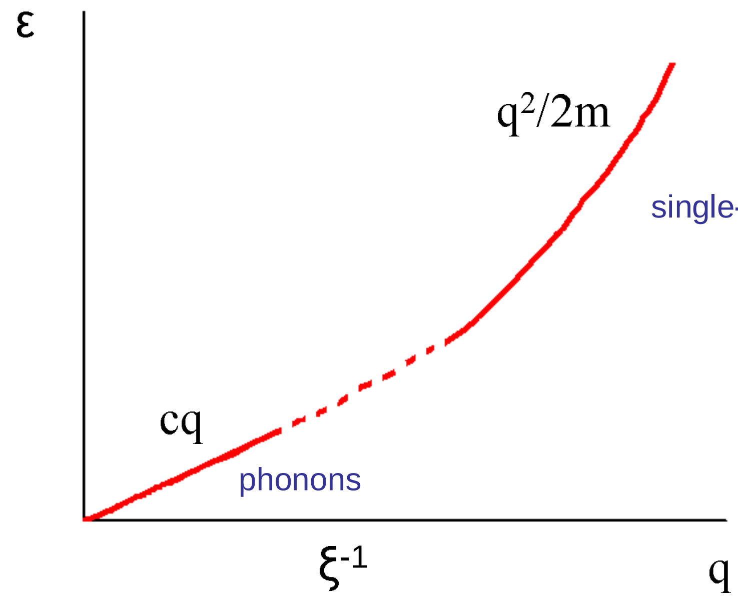
with $\boxed{c = \sqrt{g n_0 / m}}$

Wavelength of the oscillation:

$$\boxed{\lambda = 2\pi / q}$$

to be compared with the healing length

$$\boxed{\xi = \hbar / \sqrt{2mgn_0} = \hbar / \sqrt{2mc}}$$



Bogoliubov equations

Solutions of Bogoliubov equations in a uniform gas: $u, v \propto e^{i\mathbf{q}\cdot\mathbf{r}}$

Bogoliubov dispersion law

$$\omega^2 = \hbar^2 \left(\frac{q^2}{2m} \right)^2 + q^2 c^2$$

with $c = \sqrt{g n_0 / m}$

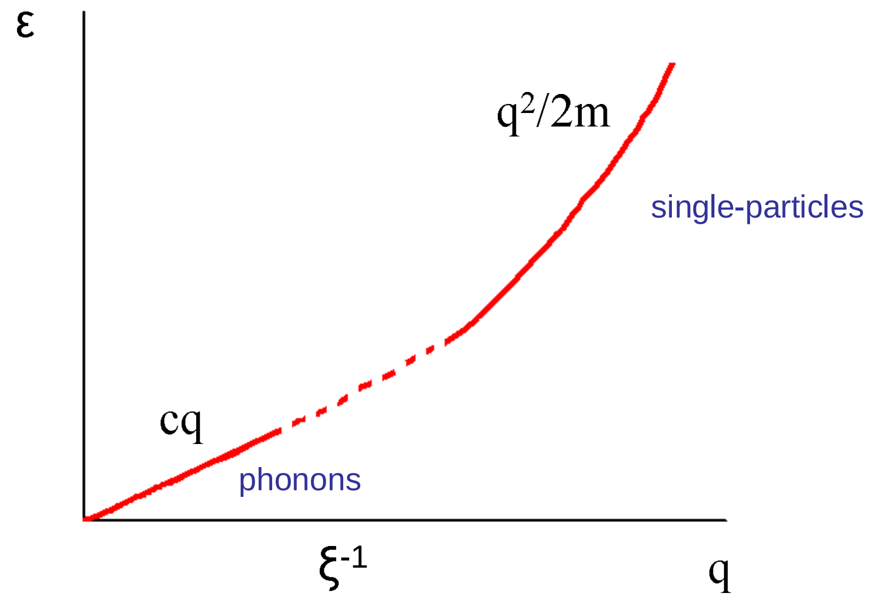
Wavelength of the oscillation:

$$\lambda = 2\pi / q$$

to be compared with the healing length

$$\xi = \hbar / \sqrt{2 m g n_0} = \hbar / \sqrt{2 m c}$$

In nonuniform systems: numerical solutions (eigenvalue problem)



Time-dependent Gross-Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t}\Psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left|\Psi_0(\mathbf{r},t)\right|^2 \right]\Psi_0(\mathbf{r},t)$$

This equation can be

- ✓ Numerically solved (GP simulations)
- ✓ Linearized for small oscillations (Bogoliubov equations)
- ✓ Rewritten in terms of density and velocity (T=0 hydrodynamics)

Time-dependent Gross-Pitaevskii equation

Rewritten in terms of density and velocity

Write $\boxed{\Psi_0 = \sqrt{n}\, e^{iS}}$ with

$$n = |\Psi_0|^2 \quad \text{density}$$

$$\mathbf{v}_S = (\hbar / m)\nabla S \quad \text{velocity}$$

and insert into

$$i\hbar \frac{\partial}{\partial t}\Psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi_0(\mathbf{r},t)|^2 \right] \Psi_0(\mathbf{r},t)$$

$$\frac{\partial}{\partial t} n + \nabla\cdot(\mathbf{v}_S n) = 0$$

$$m\frac{\partial}{\partial t}\mathbf{v}_S + \nabla\left(\frac{1}{2} m v_S^2 + V_{ext} + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n} \right) = 0$$

Time-dependent Gross-Pitaevskii equation

Rewritten in terms of density and velocity

quantum pressure

These look like hydrodynamic equations,
except for quantum pressure

$$\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v}_S n) = 0$$

$$m \frac{\partial}{\partial t} \mathbf{v}_S + \nabla \left(\frac{1}{2} m v_S^2 + V_{ext} + gn - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \right) = 0$$

Time-dependent Gross-Pitaevskii equation

What is quantum pressure in terms of energy density:

$$n = |\Psi_0|^2$$

$$\mathbf{v}_S = (\hbar / m)\nabla S$$

GP kinetic energy:

$$\boxed{E_{kin} = \frac{\hbar^2}{2m}\int d\mathbf{r}\,|\nabla\Psi_0|^2 = \frac{m}{2}\int d\mathbf{r}\, v_S^2 n + \frac{\hbar^2}{2m}\int d\mathbf{r}\left(\nabla\sqrt{n}\right)^2}$$

energy of the condensate flow

quantum pressure

To be compared with mean-field energy density $\approx gn$

Time-dependent Gross-Pitaevskii equation

$$\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v}_S n) = 0$$

$$m \frac{\partial}{\partial t} \mathbf{v}_S + \nabla \left(\frac{1}{2} m v_S^2 + V_{ext} + gn - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \right) = 0$$

Hydrodynamic equations are obtained when quantum pressure is negligible, i.e., if during the oscillation the density varies over distances λ such that

$\hbar^2 / m\lambda^2 << gn$ or $\boxed{\lambda >> \xi}$

Time-dependent Gross-Pitaevskii equation

$$\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v}_S n) = 0$$

$$m \frac{\partial}{\partial t} \mathbf{v}_S + \nabla \left(\frac{1}{2} m v_S^2 + V_{ext} + gn - \cancel{\frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n}} \right) = 0$$

$$\lambda >> \xi$$

$$\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v}_S n) = 0$$

$$m \frac{\partial}{\partial t} \mathbf{v}_S + \nabla \left(\frac{1}{2} m v_S^2 + V_{ext} + gn \right) = 0$$

Hydrodynamic equations of a superfluid at T=0

v_s is the superfluid velocity. The velocity field is irrotational!

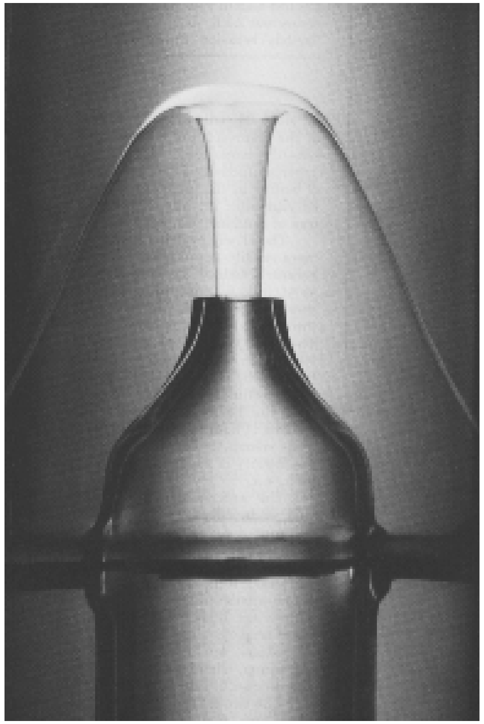
Note: **Planck constant** has disappeared !

Order parameter
and superfluidity

Superfluidity

"[…] from a modern point of view, superfluidity is
not a single phenomenon but a complex of phenomena"

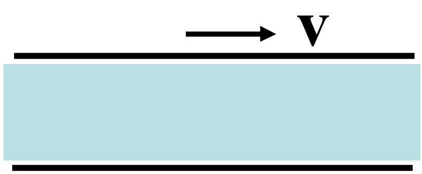
[A.J. Leggett, Rev. Mod. Phys. 73, 307 (2001)]



Superfluidity

Landau criterion for superfluidity

Case #1: Fluid at rest in the presence of moving walls or impurities, at T=0.



The dynamics in the moving frame is governed by $H = H_0 - \mathbf{v}\cdot\mathbf{p}$

The fluid can absorb energy and momentum **only** through the creation of elementary excitations. Creation of excitation costs energy.

At T=0 no excitation is created if $\varepsilon(p) - \mathbf{v}\cdot\mathbf{p} \geq 0$ $\Rightarrow$ the fluid remains at rest.

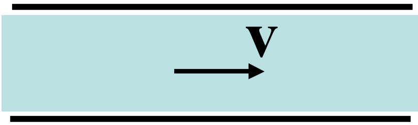
If $\boxed{v_c = \min_p \frac{\varepsilon(p)}{p} \neq 0}$ the fluid remains at rest for $v \leq v_c$

Note: the fluid at rest is **not the ground state** of H. $\Rightarrow$ **metastable state!**
The ground state would have $J = Nmv$

Superfluidity

Landau criterion for superfluidity

Case #2: Fluid moving in the presence of walls or impurities at rest, at T=0.



Similar arguments as before

Fluid current can decay **only** through the creation of elementary excitations. Creation of excitation costs energy.

if $\boxed{v_c = \min_p \frac{\varepsilon(p)}{p} \neq 0}$ the current will not decay for $v \leq v_c$ (**persistent current**)

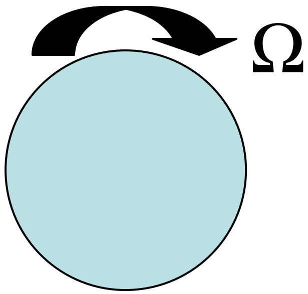
metastable state!

Note: the fluid at rest is **not the ground state** of H.
The ground state would have $J = 0$

Superfluidity

Landau criterion for superfluidity

Case #3: fluid at rest in a rotating bucket



Dynamics in rotating frame governed by $H = H_0 - \Omega \hat{l}_z$

Fluid at rest if $\Omega < \Omega_C$ where $\boxed{\Omega_C = \min_l \frac{\varepsilon(l)}{l}}$

$\varepsilon(l) \equiv$ energy of elementary excitation

$l \equiv$ angular momentum of elementary excitation

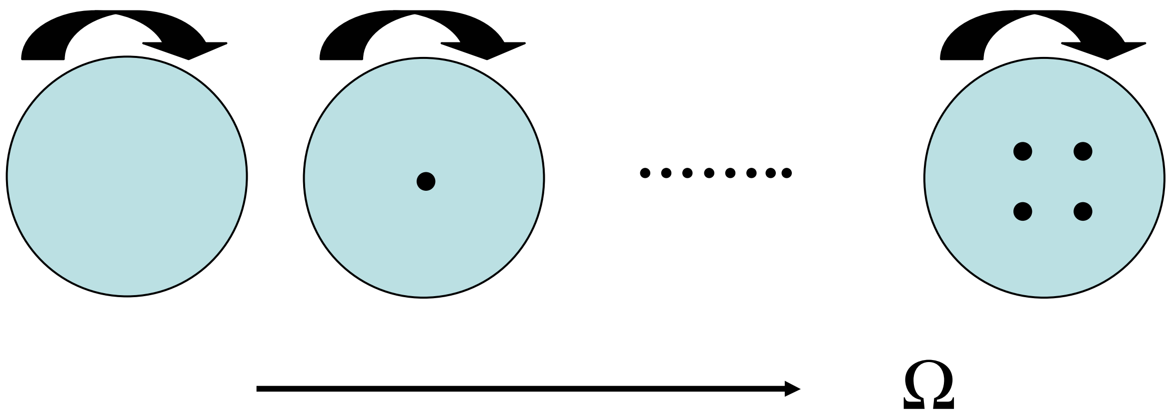
$\Omega_C \neq 0 \Longleftrightarrow$ Landau criterion for superfluidity

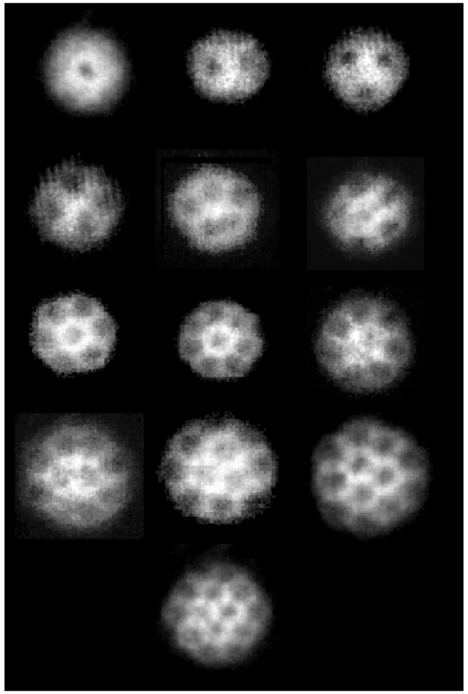
Superfluidity

Landau criterion for superfluidity

Case #3: fluid at rest in a rotating bucket

The superfluid does not follow the rotation of the bucket for small Ω, but at higher Ω it can lower its energy by nucleating vortices!





BEC and superfluidity

A **BEC** behaves as an irrotational **superfluid**, as a consequence of

$$\boxed{\Psi_0 = \sqrt{n}\, e^{iS}}$$

with

$$n = |\Psi_0|^2$$

$$\mathbf{v}_S = \hbar / m \nabla S$$

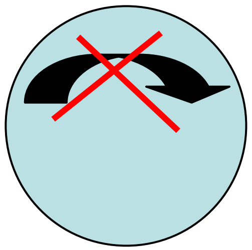
The velocity field
is the gradient of a
scalar

$$\nabla \times \mathbf{v}_S = 0$$

$$\boxed{\oint d\ell \cdot \mathbf{v}_S = 0}$$

For any closed
path in a simply
connected geometry

No rotation!



BEC and superfluidity

$$\boxed{\oint d\ell \cdot \mathbf{v}_S = 0}$$

For any closed path in a simply connected geometry

However, if the system is not simply connected (e.g., it has a hole), than one can choose a path such that

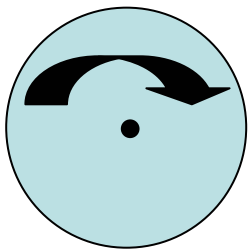
$$\boxed{\oint d\ell \cdot \mathbf{v}_S = \frac{\hbar}{m} \underbrace{\oint d\ell \cdot \nabla S}_{\Delta S = 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots} = k\frac{h}{m}}$$

quantized circulation!

This condition follows from the single-valuedness of the function

$$\Psi_0 = \sqrt{n}\, e^{iS}$$

Quantized vortex!



BEC and superfluidity

$$\boxed{\oint d\ell \cdot \mathbf{v}_S = 0}$$

For any closed path in a simply connected geometry

However, if the system is not simply connected (e.g., it has a hole), than one can choose a path such that

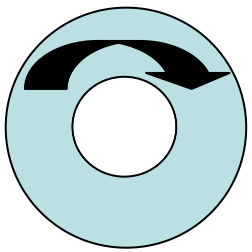
$$\boxed{\oint d\ell \cdot \mathbf{v}_S = \frac{\hbar}{m} \underbrace{\oint d\ell \, \nabla S}_{\Delta S = 2k\pi, \quad k = 0, \pm 1, \pm 2, \dots} = k \frac{h}{m}}$$

quantized circulation!

This condition follows from the single-valuedness of the function

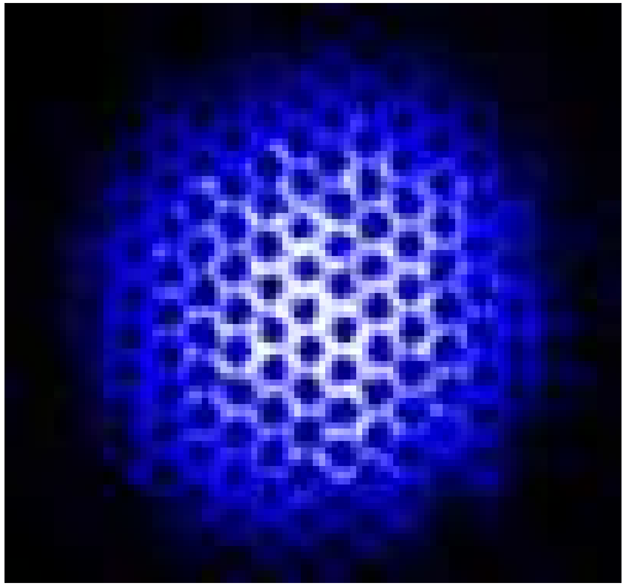
$$\Psi_0 = \sqrt{n}\, e^{iS}$$

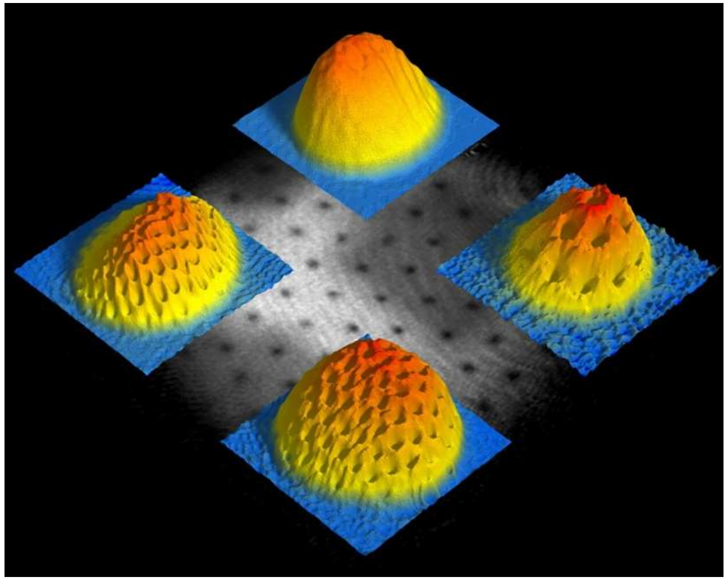
or quantized circulation in a toroidal geometry



BEC and superfluidity

Many experiments on quantized vorticity in BECs in the last decade!
A lot of interesting physics: vortex nucleation, vortex arrays, fast rotations and Lowest Landau Level regime, quantum turbulence, KT transition in 2D, Tkachenko waves, Kelvin modes, etc.





For a review, see A.Fetter, Rev. Mod. Phys. 81, 647 (2009)

Note: quantized vortices as the first clean evidence of superfluidity of fermions in the BCS-BEC crossover !

BEC and superfluid hydrodynamics

From GP equation, neglecting quantum pressure:

$$\frac{\partial}{\partial t}n + \nabla\cdot(\mathbf{v}_S n) = 0$$

$$m\frac{\partial}{\partial t}\mathbf{v}_S + \nabla\left(\frac{1}{2}mv_S^2 + V_{ext} + gn\right) = 0$$

Hydrodynamic equations of a superfluid at T=0

Can be rewritten in the form

$$\frac{\partial}{\partial t}n + \nabla\cdot(\mathbf{v}_S n) = 0$$

$$m\frac{\partial}{\partial t}\mathbf{v}_S + \nabla\left(\frac{1}{2}mv_S^2 + V_{ext} + \mu(n)\right) = 0$$

In this form they are more general !

$\longrightarrow$ this Euler equation is equivalent to the equation for the phase:

$$\hbar\frac{\partial}{\partial t}S = -\left(\frac{1}{2}mv_S^2 + V_{ext} + \mu\right)$$

local chemical potential

BEC and superfluid hydrodynamics

Hydrodynamic eqs of superfluids at T=0

$$\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v}_S n) = 0$$

$$m \frac{\partial}{\partial t} \mathbf{v}_S + \nabla \left(\frac{1}{2} m v_S^2 + V_{ext} + \mu(n) \right) = 0$$

These equations can be obtained, independently of GP, starting from the equation for the bosonic field operator in uniform systems, imposing Galilean invariance, and using a local density approximation for a slowly varying order parameter.

In this context, ***n*** is the **total** density and the superfluid velocity is

$$v_S = \frac{\hbar}{m} \nabla S$$

- ✓ Equations are **classical** (do not depend on Planck constant).
- ✓ Velocity field is **irrotational** (role of the **phase**).
- ✓ Condensate density does not enter HD eqs.
- ✓ HD valid for **macroscopic** phenomena (length scales >> healing length)
- ✓ HD applicable to both **Bose** and **Fermi** superfluids.
- ✓ HD equations depend on **equation of state** $\mu(n)$ (sensitive to quantum correlations, statistics, dimensionality, ...).
- ✓ HD equations can be linearized for small oscillations.

BEC and superfluid hydrodynamics

Hydrodynamic eqs of superfluids at T=0

$$\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v}_S n) = 0$$

$$m \frac{\partial}{\partial t} \mathbf{v}_S + \nabla \left(\frac{1}{2} m v_S^2 + V_{ext} + \mu(n) \right) = 0$$

✓ HD equations can be linearized for small oscillations.

if $n = n_{eq} + \delta n$ HD eqs become $$\frac{\partial^2}{\partial t^2} \delta n = \nabla \cdot \left[n_0 \nabla \left(\frac{\partial \mu}{\partial n} \delta n \right) \right]$$

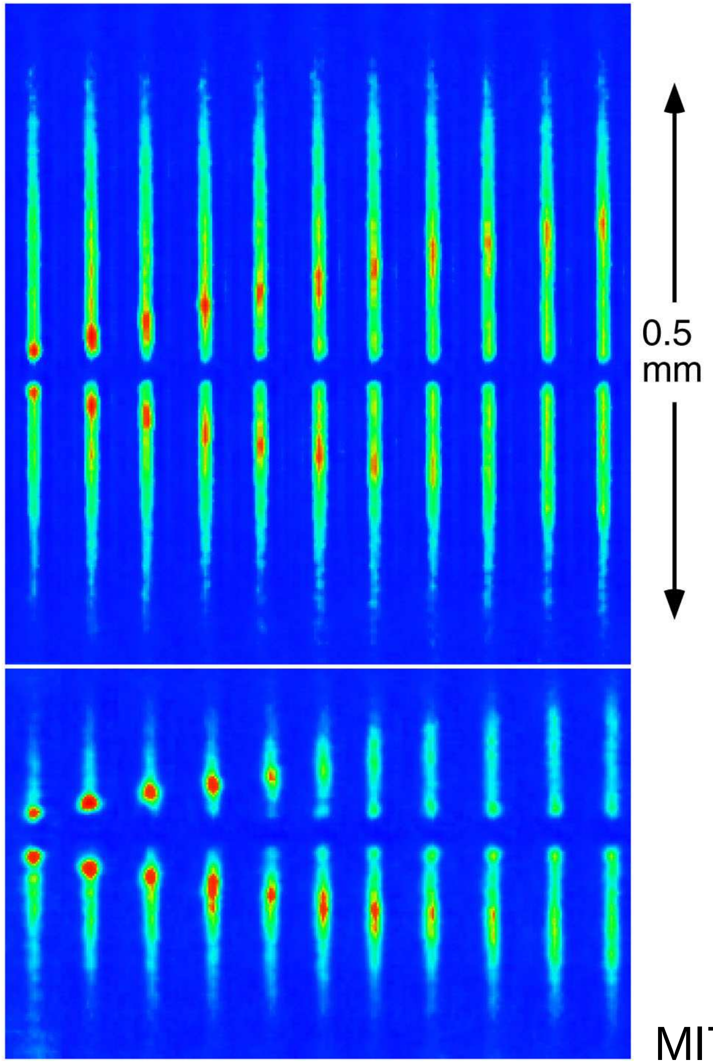
In a dilute Bose gas $\mu = gn$

and thus

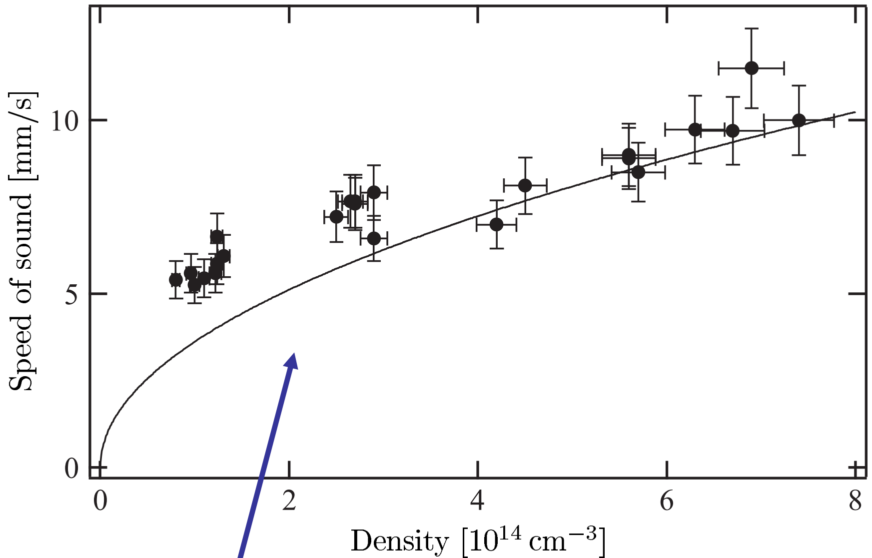
$$\frac{\partial^2}{\partial t^2} \delta n = \nabla \cdot (c^2(r) \nabla \delta n)$$

with $mc^2(r) = n\partial\mu / \partial n = \mu_0 - V_{ext}(r)$ local sound velocity

BEC and superfluid hydrodynamics



MIT, 1997



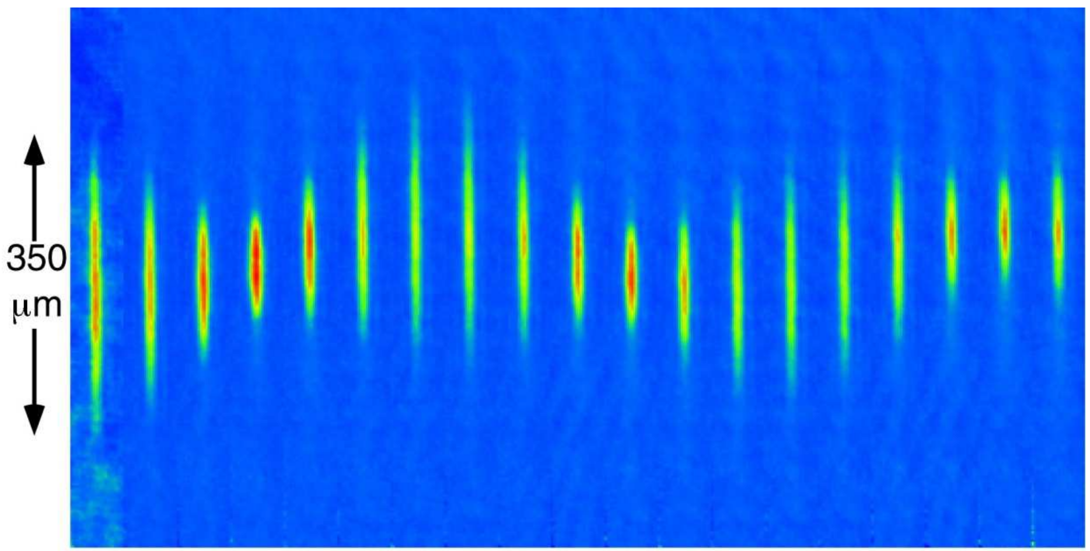
velocity of sound as a function of central density

BEC and superfluid hydrodynamics

When **wavelength** becomes comparable to the **size** of the sample the oscillations cannot be described in terms of sound waves. They involve a **motion of the whole system**.

In the presence of harmonic trapping HD equations admit simple analytic solutions for **collective excitations**.

The frequency of collective oscillations is one of observables which are measurable with the **greatest precision** in experiments with ultracold atoms!



5 milliseconds per frame

BEC and superfluid hydrodynamics

When **wavelength** becomes comparable to the **size** of the sample the oscillations cannot be described in terms of sound waves. They involve a **motion of the whole system**.

In the presence of harmonic trapping HD equations admit simple analytic solutions for **collective excitations**.

The frequency of collective oscillations is one of observables which are measurable with the **greatest precision** in experiments with ultracold atoms!

When quantum pressure cannot be ignored (small wavelength, rapidly varying potentials, soliton and vortices, etc.) the full **GP equation** can be used instead of HD equations, in dilute condensates at T=0.

If the gas is **not dilute** and/or at **finite temperature** one needs more… (see Allan's lectures).

What next:

BECs in optical lattices