Mean-field theory of trapped atomic cold gases

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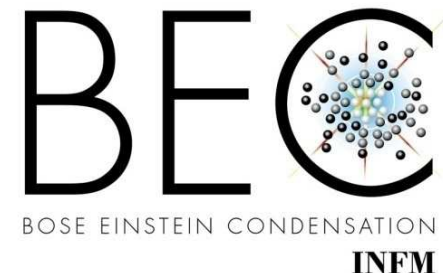
BEC
BOSE EINSTEIN CONDENSATION
INFM

UNITN: **Stringari, Pitaevskii, Dalfovo, Giorgini.**
INFM researchers: **Smerzi, Carusotto, Menotti, Recati.**
Postdocs: **Pilati, Chwedenczuk, Hyllus, Sarchi.**
Students: **Bariani, Bausmerth, Piazza, Bertaina, Larcher, Landini**
Former members: Minniti, Guilleumas, Vichi, Brunello, Falco,
Zambelli, Viverit, Weidong Li, Giorgetti, Poulsen, Pedri, Kraemer,
Ianeselli, Trefzger, Jackson, Cozzini, Tozzo, Astrakharchik,
Wouters, Idziaszek, Orso, Tsuchiya, Calarco, Lobo, De Chiara,
Furlan, Antezza, Pezzè, Watanabe, Taylor.
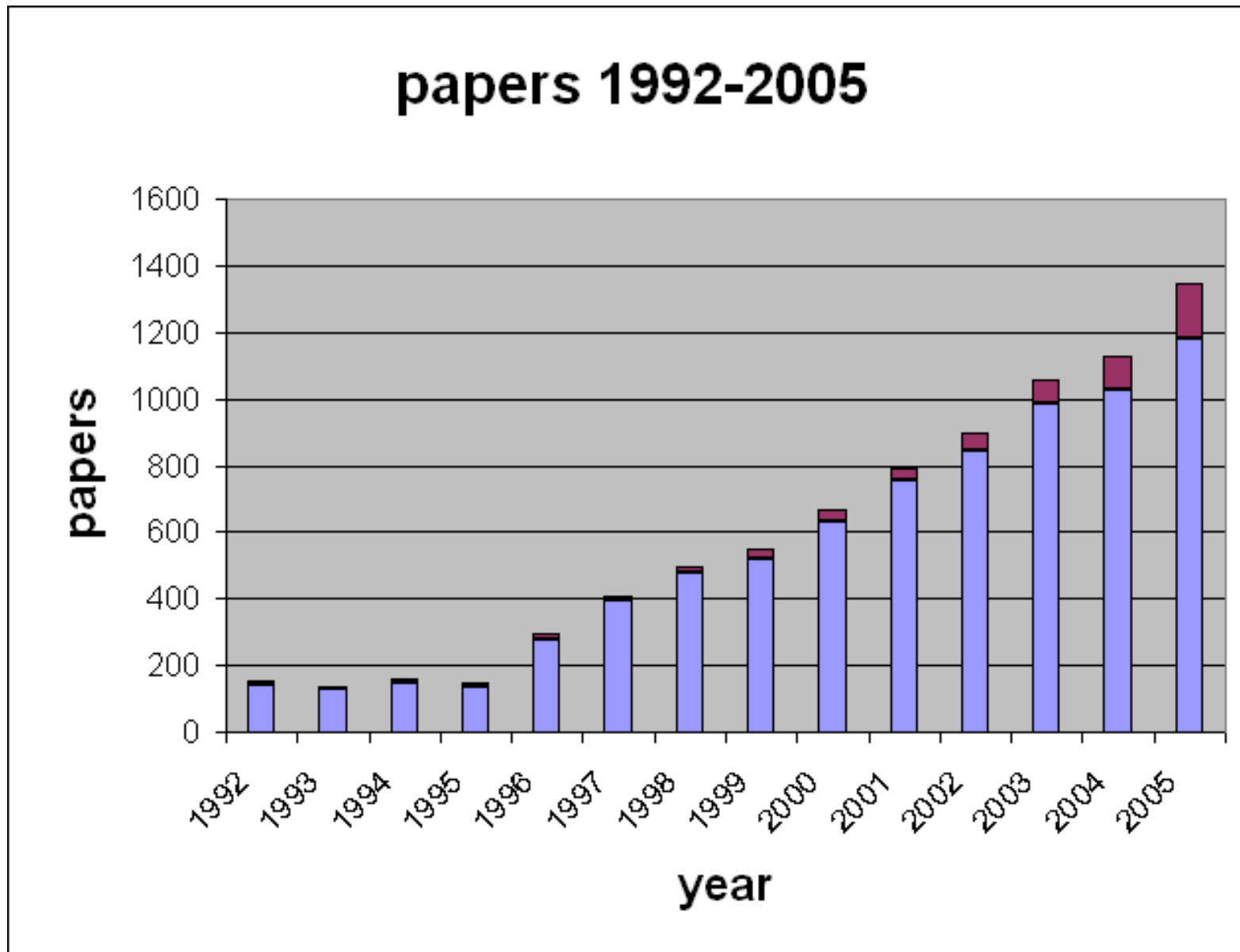
http://bec.science.unitn.it

Prelude

First images of BEC with Rb atoms (JILA 1995)
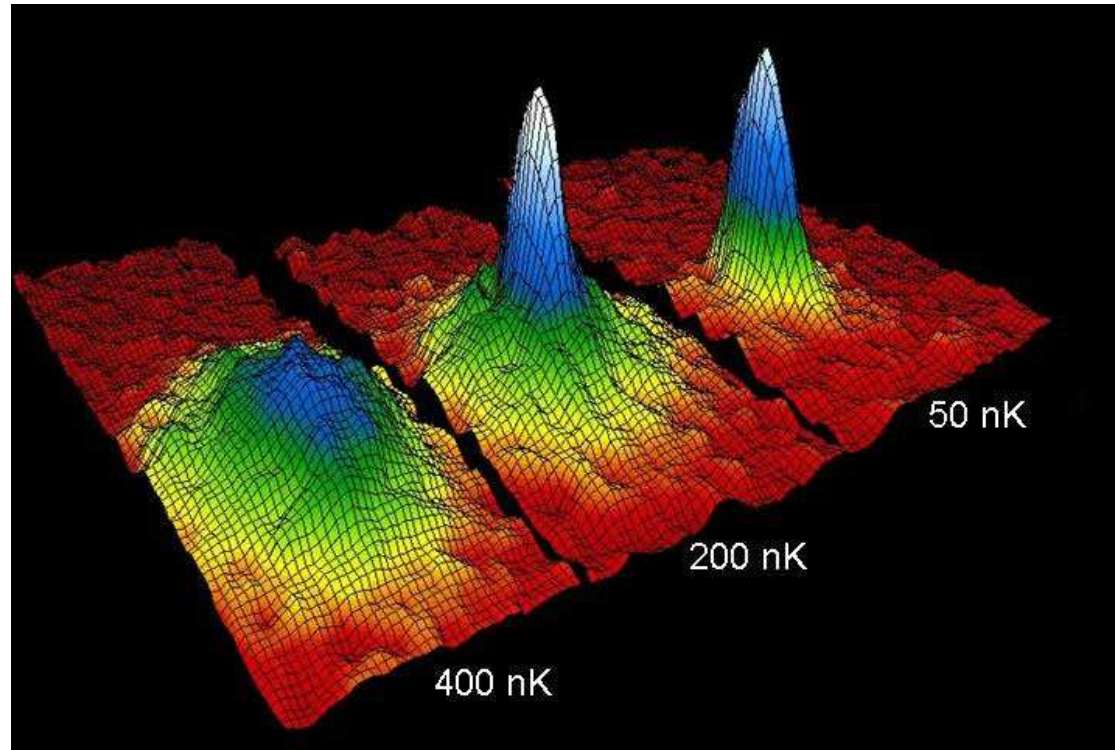
At present: many condensates with more atomic species (Rb, Na, Li, H, He*, K, Cs, Yb), including condensates of molecules (Rb_2, Na_2, Cs_2) and also degenerate fermions, fermionic superfluidity, BCS-BEC crossover, fermi-fermi, bose-fermi, bose-bose mixtures, etc.

papers 1992-2005

Source: ISI – Web of Science
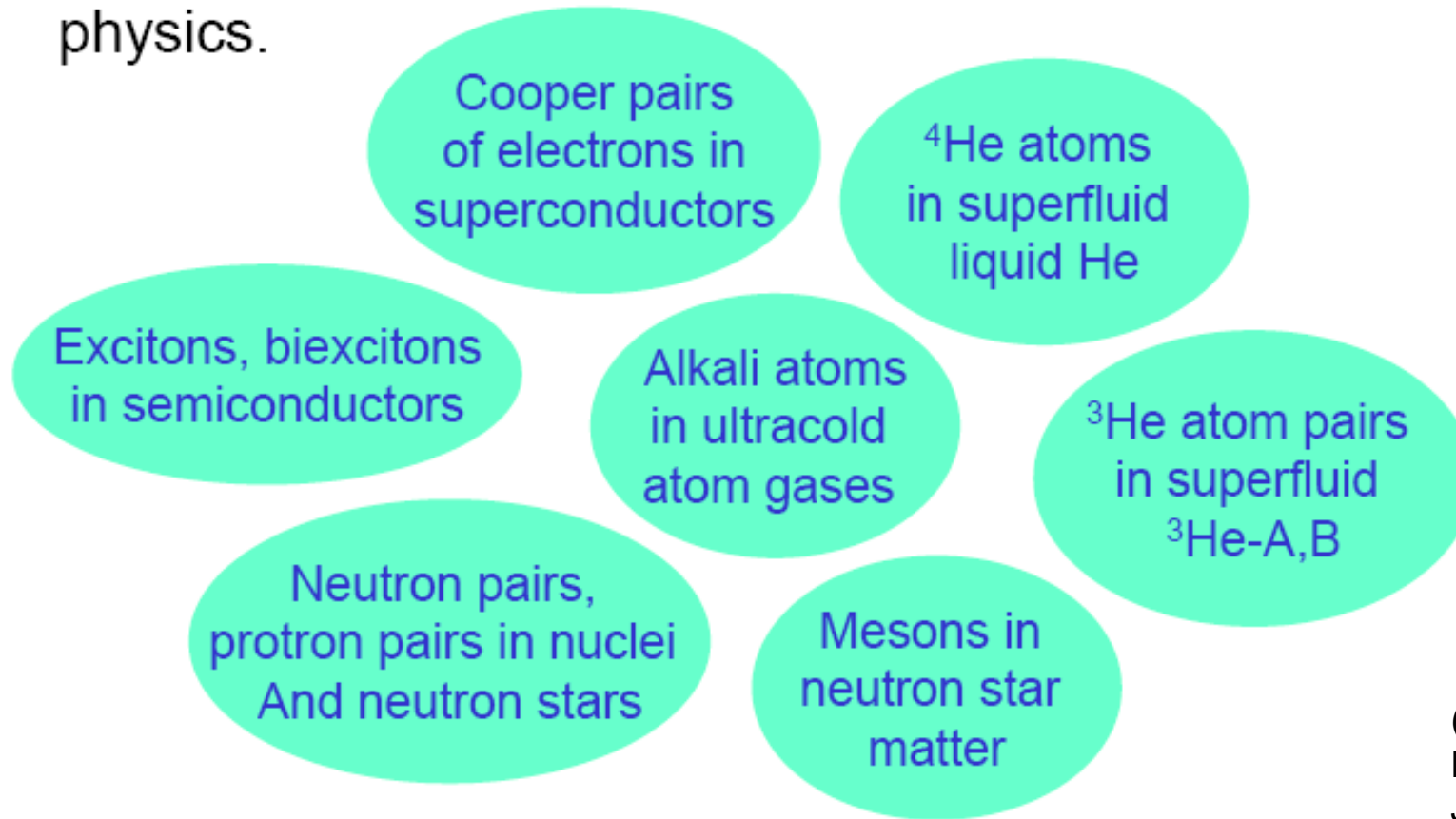Keywords: bos* AND condens* (blue); cold OR ultracold AND fermi* (red)

Why BEC is important ?

Paradigm of statistical mechanics (phase transition in the absence of interactions).

Exact description of the effects of interactions for dilute gases.

Fundamental concepts (long range order; spontaneous symmetry breaking; order parameter, quasi-particles, etc.) which play an important role in many areas of physics.

BEC shows up in condensed matter, nuclear physics, elementary particle physics, astrophysics, and atomic physics.

Cooper pairs of electrons in superconductors

^{4}He atoms in superfluid liquid He

Excitons, biexcitons in semiconductors

Alkali atoms in ultracold atom gases

^{3}He atom pairs in superfluid ^{3}He-A,B

Neutron pairs, protron pairs in nuclei And neutron stars

Mesons in neutron star matter

(taken from Debbie Jin, JILA)

Most recent: **exciton-polariton gas in 2D cavities in semiconductors**

Some relevant properties of BEC

Those due to interaction:
- **sound propagation and collective oscillations**
- **solitary waves**

Those due to phase coherence:
- **interference**
- **atom laser**

Superfluid properties (interaction + coherence):
- **viscousless motion**
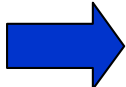- **quantized vortices**
- **Josephson effect**

Quantum phase transitions:
- **Superfluid - Mott insulator**
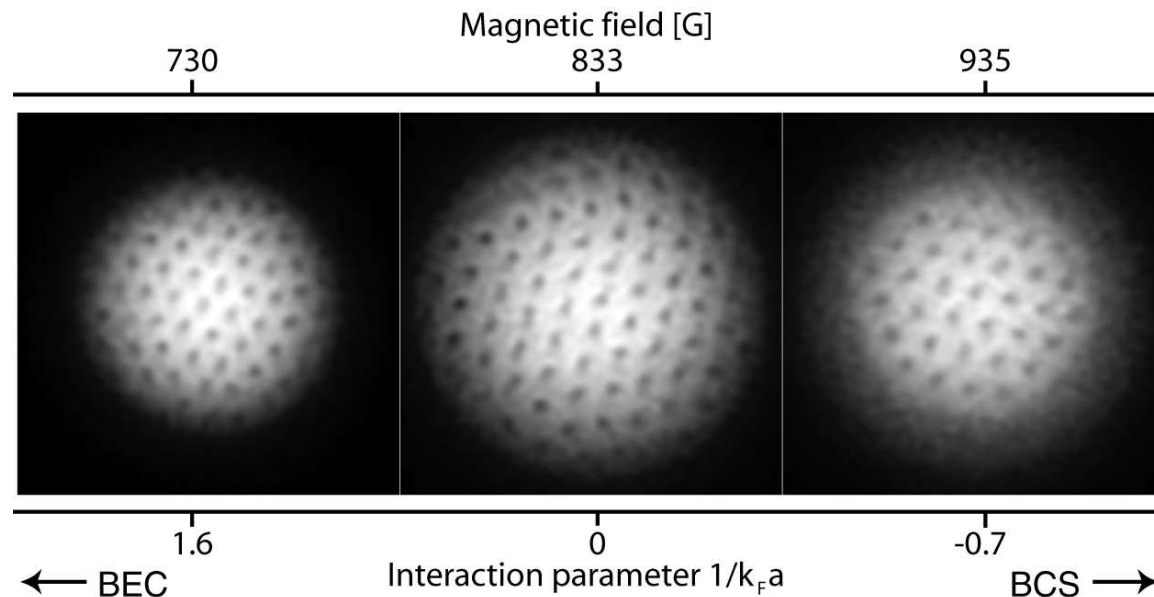- **Kosterlitz-Thouless**

Changing the statistics and tuning the interaction

Cooling fermions to degeneracy.

Tuning the interaction and/or the external confining potentials:

from single-particle "textbook" physics ➡️ to correlated many-body physics

Pairing of fermions: from BCS to molecular BEC

Why mean-field theories are important ?

These ultracold gases are dilute.

Exact results can be obtained by including the interaction at the mean-field level.

Many properties of dilute BECs are well described by the Gross-Pitaevskii theory.

Accurate mean-field theories are also available for fermions.

A quantitative comparison between theory and experiment is possible at the mean-field level and beyond !

Bibliography:

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"Theory of Bose-Einstein condensation in trapped gases",
Rev. Mod. Phys . **71** , 463 (1999)

S.Giorgini, L.Pitaevskii, and S.Stringari,
"Theory of ultracold atomic Fermi gases",
Rev. Mod. Phys. **80**, 1215 (2008)

A.Leggett ,
"Bose-Einstein condensation in the alkali gases: Some fundamental concepts"
Rev. Mod. Phys . **73** , 307 (2001)
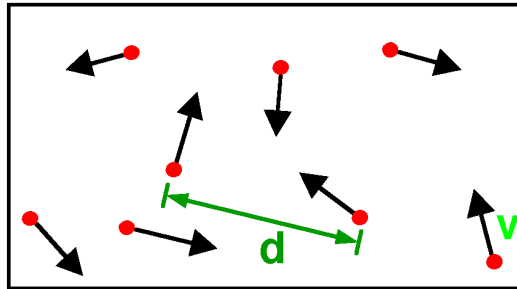
Bose-Einstein Condensation
Edited by A.Griffin, D.W.Snoke and S.Stringari
Cambridge University Press (13 Jul 1996)

C. Pethick and H. Smith ,
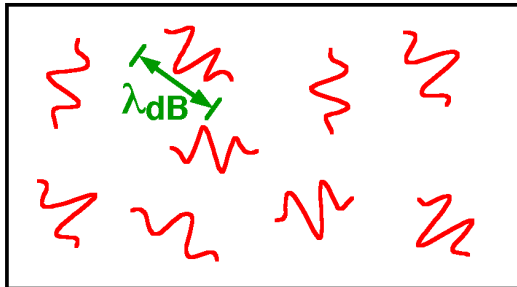*"Bose Einstein Condensation in Dilute Bose Gases"*,
Cambridge University Press (2002)

L. Pitaevskii and S. Stringari,
"Bose Einstein Condensation",
Oxford University Press (2003)

Bose-Einstein Condensation

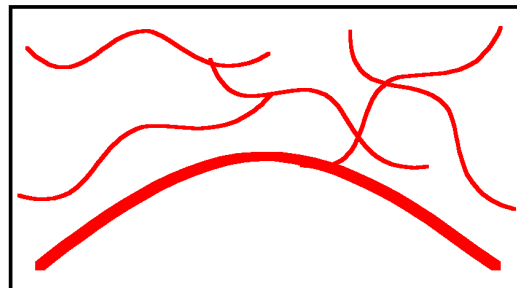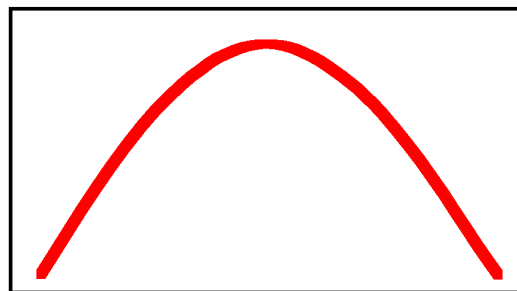# What is Bose-Einstein condensation (BEC)?

High Temperature T:

thermal velocity v
density d^{-3}

"Billiard balls"

Low Temperature T:

De Broglie wavelength
$\lambda_{dB} = h/mv \propto T^{-1/2}$

"Wave packets"

T=T_{crit}:
Bose-Einstein Condensation

$\lambda_{dB} \approx d$

"Matter wave overlap"

T=0:
Pure Bose condensate

"Giant matter wave"

(Taken from W. Ketterle)

BEC in noninteracting gases

Textbook problem of quantum statistical mechanics.
In the grandcanonical ensemble:

Occupation number of single-particle states:

$$n_i = \frac{1}{\exp[(\varepsilon_i - \mu)/k_B T] - 1}$$

where

$$H\varphi_i = \varepsilon_i \varphi_i$$

The value of μ is fixed by normalization condition

$$\sum_i n_i = N$$

BEC starts when the chemical potential is so close to ε_0 that $(\varepsilon_0 - \mu) \ll k_B T$ and the occupation number of $i=0$ state ($n_0 \equiv N_0$) becomes large and comparable to N:

$$N_0 \approx \frac{k_B T}{\varepsilon_0 - \mu} \gg 1$$

$$\underline{\hspace{2cm}}\ \varepsilon_1$$

$$\underline{\hspace{2cm}}\ \varepsilon_0$$

$$\mu\ \underline{\hspace{2cm}}$$

BEC in noninteracting gases

Occupation number of single-particle states:

$$n_i = \frac{1}{\exp[(\varepsilon_i - \mu)/k_B T] - 1}$$
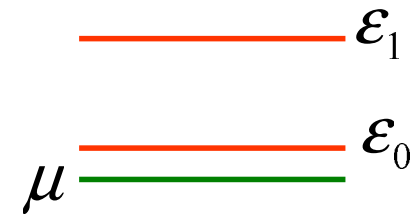
where

$$H\varphi_i = \varepsilon_i \varphi_i$$

The value of μ is fixed by normalization condition

$$\sum_i n_i = N$$

If $\varepsilon_i - \mu \gg \varepsilon_0 - \mu$

for *i>0,* one can replace μ with ε_0 in the above expression and the occupation number of i-state **does not depend** any more on **N !**
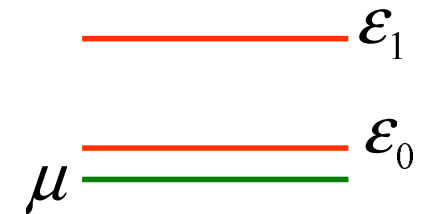
Mechanism of BEC:

$$N = N_0 + \sum_{i \neq 0} \frac{1}{\exp[(\varepsilon_i - \varepsilon_0)/k_B T] - 1}$$

$$\varepsilon_1$$

$$\varepsilon_0$$

$$\mu$$

number of atoms N_T out of the condensate depends only on T, not on N.

The condition **N_T=N** fixes the value of critical temperature

BEC in noninteracting gases in 3D harmonic potential
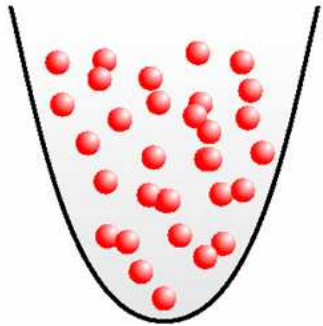
$$H\varphi_i = \varepsilon_i \varphi_i$$

Single-particle hamiltonian: $H = p^2/2m + V_{ext}$

Confining potential:
$$V_{ext} = \frac{1}{2} m \left[\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right]$$
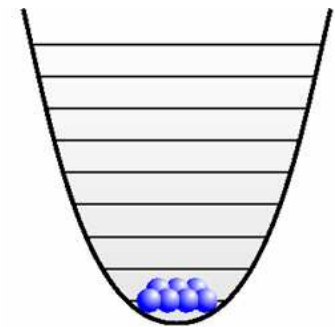
Spectrum of eigenstates:

$$\varepsilon(n_x, n_y, n_z) = (n_x + \frac{1}{2})\hbar\omega_x + (n_y + \frac{1}{2})\hbar\omega_y + (n_z + \frac{1}{2})\hbar\omega_z$$

Condition for BEC:

$$\mu = \varepsilon(0,0,0)$$

$$N_T = \sum_{n_x, n_y, n_z \neq 0} \frac{1}{\exp[\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)] - 1}$$

BEC in noninteracting gases in 3D harmonic potential
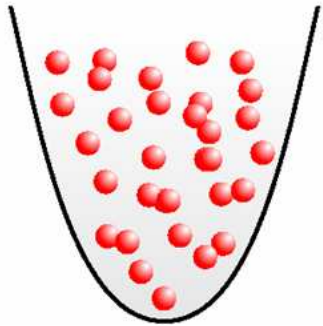
$$N_T = \sum_{n_x, n_y, n_z \neq 0} \frac{1}{\exp[\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)] - 1}$$

If $k_B T \gg \hbar\omega_i$

then one can transform the discrete sum into an integral (semiclassical approximation).

The integration gives

$$N_T = \left(\frac{k_B T}{\hbar\omega_{ho}}\right)^3 g_3(1)$$

where $\omega_{ho} = (\omega_x \omega_y \omega_z)^{1/3}$

and $g_3(1) = \varsigma(3)$, with $\varsigma(n)$ Riemann ς function

Critical temperature when $N_T = N$, which implies:

$$k_B T_c = 0.94 \hbar\omega_{ho} N^{1/3}$$

and

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^3$$

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^3$$

Condensate fraction measured at JILA (1996)
EXPERIMENTAL EVIDENCE
OF PHASE TRANSITION

BEC in interacting gases

Many-body Hamiltonian:

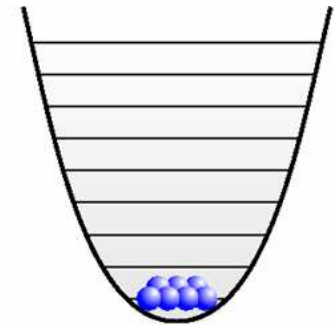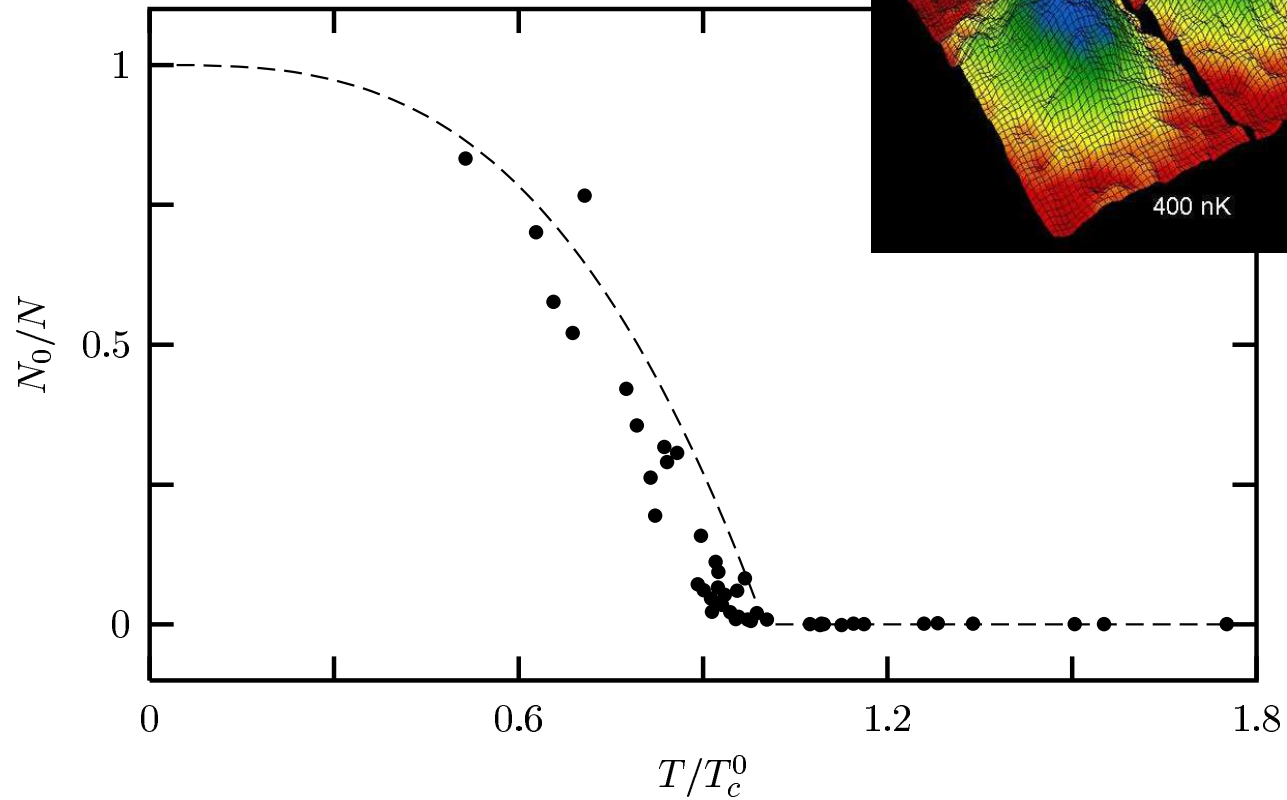$$\hat{H} = \int d\mathbf{r}\, \hat{\Psi}^+(\mathbf{r}) \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}) + \frac{1}{2} \iint d\mathbf{r}\, d\mathbf{r}'\, \hat{\Psi}^+(\mathbf{r}) \hat{\Psi}^+(\mathbf{r}') V(|\mathbf{r} - \mathbf{r}'|) \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r})$$

where $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^+(\mathbf{r})$ are bosonic field operators.

one-body density matrix: $\qquad n^{(1)}(\mathbf{r}, \mathbf{r}') = \left\langle \hat{\Psi}^+(\mathbf{r}) \hat{\Psi}(\mathbf{r}') \right\rangle$

Relevant observables directly related to the one-body density matrix:

particle density: $\quad n(\mathbf{r}) = \left\langle \hat{\Psi}^+(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right\rangle = n^{(1)}(\mathbf{r}, \mathbf{r})$

momentum distribution:

$$n(\mathbf{p}) = \left\langle \hat{\Psi}^+(\mathbf{p}) \hat{\Psi}(\mathbf{p}) \right\rangle = (2\pi\hbar)^{-3} \int d\mathbf{R}\, d\mathbf{s}\, n^{(1)}(\mathbf{R} + \mathbf{s}/2, \mathbf{R} - \mathbf{s}/2) e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar}$$

BEC in interacting gases

momentum distribution:

$$n(\mathbf{p}) = (2\pi\hbar)^{-3} \int d\mathbf{R}\, d\mathbf{s}\ n^{(1)}(\mathbf{R} + \mathbf{s}/2, \mathbf{R} - \mathbf{s}/2)\, e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar}$$

In uniform systems
$$n^{(1)}(\mathbf{r}, \mathbf{r}') = n^{(1)}(s) = \frac{1}{V} \int dp\, n(p)\, e^{ips/\hbar}$$

Usual situation: *n(p)* is a smooth function of p at low p
$\Longrightarrow$
$$n^{(1)}(s)_{s \to \infty} = 0$$

However, the occurrence of BEC in noninteracting gases suggests that the *p=0* state (lowest single-particle state in the uniform system) can be macroscopically occupied below a given critical temperature T_c.
In terms of momentum distribution this means:

$$n(p) = N_0 \delta(p) + \tilde{n}(p)$$

BEC   **Smooth function**

BEC in interacting gases

$$n(p) = N_0 \delta(p) + \tilde{n}(p)$$ $\Longrightarrow$ $$n^{(1)}(s)_{s \to \infty} = n_0 = \frac{N_0}{V}$$

Off-diagonal long range order
(Landau, Lifschitz, Penrose, Onsager)

Example of calculation
of density matrix in highly
correlated many-body
system: liquid He4

(Ceperley, Pollock 1987)

$T \ll T_C$

$T > T_C$

condensate
fraction
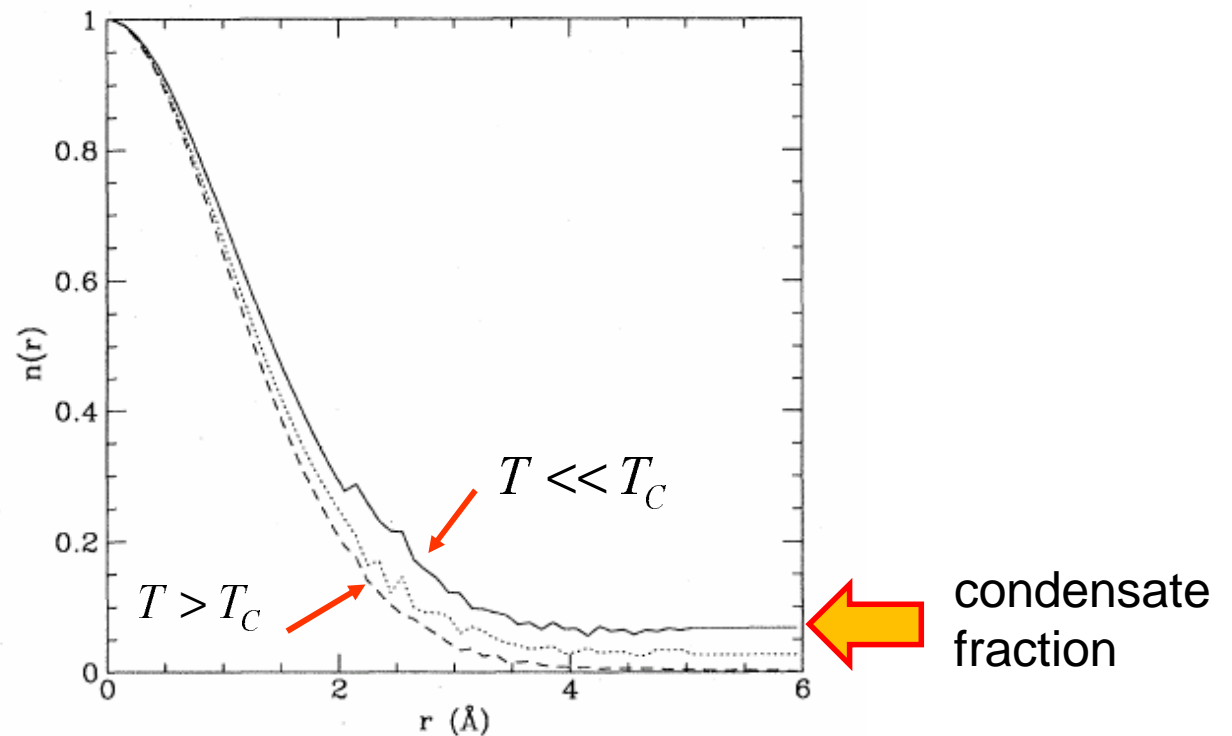
BEC in interacting gases

Same concept, but in a more general form, including finite nonuniform systems. Starting from the eigenvalues and eigenfunctions of the one-body density matrix.

One can define the eigenfunctions of the density matrix through this relation:

$$\int d\mathbf{r}'\, n^{(1)}(\mathbf{r},\mathbf{r}')\varphi_i(\mathbf{r}') = n_i\,\varphi_i(\mathbf{r})$$

Once the orthonormalized eigenfunctions are found, one can write

$$n^{(1)}(\mathbf{r},\mathbf{r}') = \sum_i\, n_i\,\varphi_i^*(\mathbf{r})\varphi_i(\mathbf{r}')$$

Single-particle occupation numbers

BEC in interacting gases

$$\int d\mathbf{r}' \, n^{(1)}(\mathbf{r},\mathbf{r}')\varphi_i(\mathbf{r}') = n_i \, \varphi_i(\mathbf{r})$$

$$n^{(1)}(\mathbf{r},\mathbf{r}') = \sum_i \, n_i \varphi_i^*(\mathbf{r})\varphi_i(\mathbf{r}')$$

BEC occurs when $\quad n_o \equiv N_0 \gg 1$

If this happens, then it is convenient to rewrite the density matrix
by separating the contribution arising from the condensate:

$$n^{(1)}(\mathbf{r},\mathbf{r}') = N_0 \varphi_0^*(\mathbf{r})\varphi_0(\mathbf{r}') + \sum_{i \neq 0} n_i \, \varphi_i^*(\mathbf{r})\varphi_i(\mathbf{r}')$$

For large N the sum tends to zero at large distances.
Conversely, the first term remains finite even at large |*r-r'*|, where one recovers the
concept of long range order.
But the diagonalization of the density matrix works even for finite systems, where
BEC is identified with the eigenfunction having the largest eigenvalue (of the order
of N).

BEC in interacting gases

Example of diagonalization of the one-body density matrix: He4 droplets.

[Lewart, Pandharipande and Pieper, Phys. Rev. B (1988)]

TABLE III. Number of particles having a given angular momentum for the 70-atom ^{4}He drop. The number of particles in the condensate is shown separately.

l	N_l
Condensate	25.33 ⎫ 26.46
0	1.13 ⎭
1	3.74
2	5.06
3	5.51
4	5.55
5	4.92
6	4.20
7	3.63
8	2.98
9	1.97
10	1.32
Total	65.34
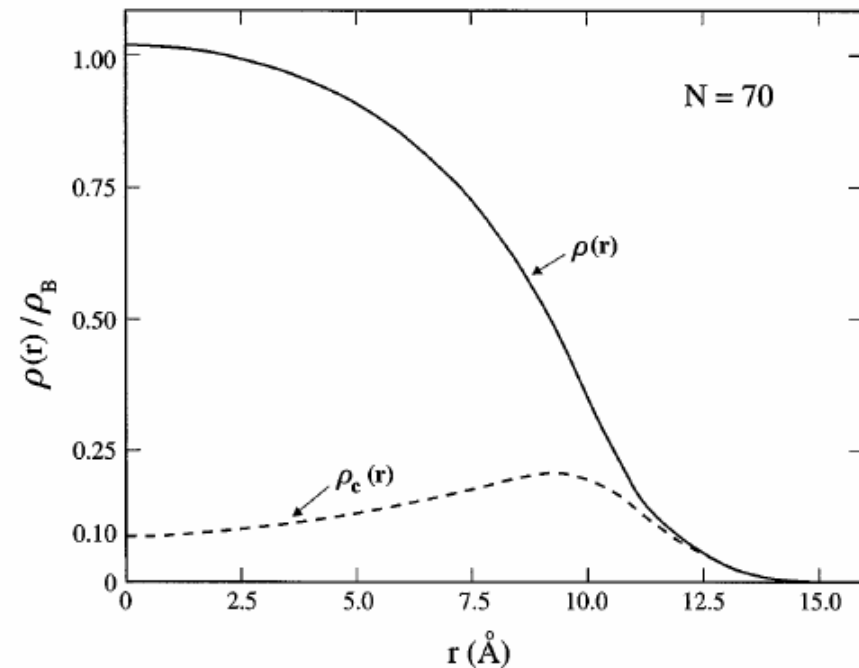
Note: In bulk superfluid helium the condensate fraction is of the order of 0.1. In the droplet, the condensate fractions is locally larger near the surface, where the system is more dilute.

BEC in interacting gases

$$\int d\mathbf{r}' \, n^{(1)}(\mathbf{r}, \mathbf{r}') \varphi_i(\mathbf{r}') = n_i \, \varphi_i(\mathbf{r})$$

$$n^{(1)}(\mathbf{r}, \mathbf{r}') = \sum_i n_i \, \varphi_i^*(\mathbf{r}) \varphi_i(\mathbf{r}')$$

BEC occurs when $n_o \equiv N_0 \gg 1$

Single-particle occupation numbers

"We shall say that at any given time *t*, the system shows **BEC** if one or more of the eigenvalues $n_i(t)$ is of the order of the total number of particles N; and further that it shows **simple BEC** if one and only one eigenvalue is of order N, all the rest being of order 1. Systems showing nonsimple BEC (i.e., having more than one eigenvalue of order N) are sometimes said to be **fragmented**. The phrase "of order N(1)" is somewhat vague in a situation where there is no simple thermodynamic limit, but in practice this does not usually lead to difficulty."
[A.J. Leggett, Rev. Mod. Phys. 73, 307 (2001)]

Order parameter and Gross-Pitaevskii equation

Order parameter

Starting from the definition of $n^{(1)}$

$$n^{(1)}(\mathbf{r},\mathbf{r}') = \left\langle \hat{\Psi}^+(\mathbf{r})\hat{\Psi}(\mathbf{r}') \right\rangle$$

and using its eigenfunctions

$$\int d\mathbf{r}'\, n^{(1)}(\mathbf{r},\mathbf{r}')\varphi_i(\mathbf{r}') = n_i\varphi_i(\mathbf{r}) \qquad n^{(1)}(\mathbf{r},\mathbf{r}') = \sum_i n_i \varphi_i^*(\mathbf{r})\varphi_i(\mathbf{r}')$$

one can define annihilation and creation operators $\hat{a}_i, \hat{a}_i^+$

such that $\quad [\hat{a}_i, \hat{a}_j^+] = \delta_{ij} \qquad [\hat{a}_i, \hat{a}_j] = 0 \qquad \left\langle \hat{a}_j^+\hat{a}_i \right\rangle = \delta_{ij}n_i$

and $\quad \hat{\Psi}(\mathbf{r}) = \sum_i \varphi_i(\mathbf{r})\hat{a}_i \qquad \hat{\Psi}^+(\mathbf{r}) = \sum_i \varphi_i^*(\mathbf{r})\hat{a}_i^+$

Order parameter

Separating the condensate in $n^{(1)}$

$$n^{(1)}(\mathbf{r},\mathbf{r}') = N_0 \varphi_0^*(\mathbf{r})\varphi_0(\mathbf{r}') + \sum_{i \neq 0} n_i \varphi_i^*(\mathbf{r})\varphi_i(\mathbf{r}')$$

is equivalent to writing

$$\hat{\Psi}(\mathbf{r}) = \varphi_0(\mathbf{r})\hat{a}_0 + \sum_{i \neq 0} \varphi_i(\mathbf{r})\hat{a}_i$$

into the definition $\quad n^{(1)}(\mathbf{r},\mathbf{r}') = \left\langle \hat{\Psi}^+(\mathbf{r})\hat{\Psi}(\mathbf{r}') \right\rangle$

Very crucial point !
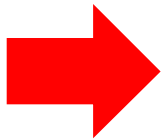
If the system exhibits BEC, then $\quad \left\langle \hat{a}_0^+ \hat{a}_0 \right\rangle = n_0 = N_0 \gg 1$

while $\quad [\hat{a}_0, \hat{a}_0^+] = 1$

This means that the noncommutativity of these operators is inessential for most physical properties.

$\hat{a}_0, \hat{a}_0^+ \quad$ can be replaced by *c*-numbers $\rightarrow \sqrt{N_0}$

Order parameter

This
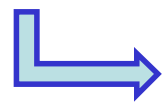$$\hat{\Psi}(\mathbf{r}) = \varphi_0(\mathbf{r})\hat{a}_0 + \sum_{i \neq 0} \varphi_i(\mathbf{r})\hat{a}_i$$

becomes
$$\hat{\Psi}(\mathbf{r}) = \Psi_0(\mathbf{r}) + \delta\hat{\Psi}(\mathbf{r})$$

with
$$\Psi_0(\mathbf{r}) = \sqrt{N_0}\,\varphi_0(\mathbf{r})$$ Condensate order parameter (function)

$$\delta\hat{\Psi}(\mathbf{r}) = \sum_{i \neq 0} \varphi_i(\mathbf{r})\hat{a}_i$$ non-condensed part (operator)

Usually fluctuations are small in dilute gases at T=0.
This implies that the **field operator** can be approximated with a **classical field**
(Bogoliubov approximation).

Order parameter

the order parameter $\Psi_0(\mathbf{r}) = \sqrt{N_0}\,\varphi_0(\mathbf{r})$

can be written as $\Psi_0(\mathbf{r}) = |\Psi_0(\mathbf{r})|\,e^{iS(\mathbf{r})}$

It is a complex function, defined up to a constant phase factor.
Fixing the phase S means breaking of gauge symmetry.

It corresponds also to write $\langle \hat{\Psi}(\mathbf{r}) \rangle = \Psi_0(\mathbf{r}) \neq 0$

where the average can be read as $\langle \hat{\Psi}(\mathbf{r}) \rangle = \langle N | \hat{\Psi}(\mathbf{r}) | N+1 \rangle$

For stationary states one has $|N(t)\rangle = e^{-iE(N)t/\hbar}|N\rangle$

And hence the stationary order parameter has this peculiar time-dependence:

$$\Psi_0(\mathbf{r},t) = e^{-i\mu t/\hbar}\,\Psi_0(\mathbf{r})$$

The key quantity is the chemical potential $\mu = E(N) - E(N-1)$

Equation for the order parameter

Is the order parameter

$$\Psi_0(\mathbf{r}, t) = \left|\Psi_0(\mathbf{r}, t)\right| e^{iS(\mathbf{r}, t)}$$

a solution of a Schrödinger-like equation **?**

Starting point: equation of motion for the field operator

$$\left\langle i\hbar \frac{\partial}{\partial t} \hat{\Psi} \right\rangle = \left\langle [\hat{\Psi}, \hat{H}] \right\rangle$$

$$\left\langle i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) \right\rangle = \left\langle \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + \int d\mathbf{r}' \hat{\Psi}^+(\mathbf{r}', t) V(|\mathbf{r} - \mathbf{r}'|) \hat{\Psi}(\mathbf{r}', t) \right] \hat{\Psi}(\mathbf{r}, t) \right\rangle$$

Equation for the order parameter

$$\left\langle i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r},t) \right\rangle = \left\langle \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + \int d\mathbf{r}' \hat{\Psi}^+(\mathbf{r}',t) V(|\mathbf{r}-\mathbf{r}'|) \hat{\Psi}(\mathbf{r}',t) \right] \hat{\Psi}(\mathbf{r},t) \right\rangle$$

This is for the field operator. But we want something for the classical field

$$\Psi_0(\mathbf{r},t) = \left\langle \hat{\Psi}(\mathbf{r},t) \right\rangle$$

Delicate issue: replacing the field operator with the classical field (order parameter) in the interaction term requires a proper procedure.

Equation for the order parameter

$$\left\langle i\hbar \frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t)\right\rangle = \left\langle \left[-\frac{\hbar^2\nabla^2}{2m}+V_{ext}(\mathbf{r})+\int d\mathbf{r}'\,\hat{\Psi}^{+}(\mathbf{r}',t)V(|\mathbf{r}-\mathbf{r}'|)\hat{\Psi}(\mathbf{r}',t)\right]\hat{\Psi}(\mathbf{r},t)\right\rangle$$

A simple and correct procedure is applicable (in 3D) if

➡ The **range** of interaction and **s-wave scattering length** *a* are much **smaller** than the average **distance d** between particles

➡ **Temperature** is sufficiently **low.**

➡ **Macroscopic variations** of the order parameter are considered (variation along distances much larger than *a*).

Only low energy two-body scattering properties are relevant for describing the many-body problem.

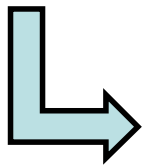# Equation for the order parameter

$$\left\langle i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r},t) \right\rangle = \left\langle \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + \int d\mathbf{r}' \hat{\Psi}^+(\mathbf{r}',t) V(|\mathbf{r}-\mathbf{r}'|) \hat{\Psi}(\mathbf{r}',t) \right] \hat{\Psi}(\mathbf{r},t) \right\rangle$$

Only low energy two-body scattering properties are relevant for describing the many-body problem.

The scattering length **a** is the only relevant interaction parameter.

The equation for order parameter is obtained by using

$$\Psi_0(\mathbf{r},t) = \left\langle \hat{\Psi}(\mathbf{r},t) \right\rangle$$

and by replacing V with the **effective potential**

$$V_{eff} = g\delta(r-r') \qquad \text{with} \qquad g = 4\pi\hbar^2 a / m$$

Equation for the order parameter

$$i\hbar \frac{\partial}{\partial t} \Psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left|\Psi_0(\mathbf{r},t)\right|^2 \right] \Psi_0(\mathbf{r},t)$$

Gross-Pitaevskii equation

density: $n\,(\mathbf{r},t) = \left|\Psi_0(\mathbf{r},t)\right|^2$

It requires that:

✓ N is large

✓ The gas is dilute (quantum fluctuations are negligible) $\longrightarrow$ $na^3 \ll 1$

✓ The temperature is low (thermal fluctuations are negligible) $\longrightarrow$ $T \ll T_C$

It is equivalent to:

✓ treating the field operator like a classical field $\hat{\Psi}(\mathbf{r}) = \Psi_0(\mathbf{r}) + \delta\hat{\Psi}(\mathbf{r})$

Equation for the order parameter

$$i\hbar \frac{\partial}{\partial t} \Psi_0(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g \left| \Psi_0(\mathbf{r}, t) \right|^2 \right] \Psi_0(\mathbf{r}, t)$$

Gross-Pitaevskii equation

For non dilute gases and/or finite T one has

$$n(\mathbf{r}, t) \neq n_0(\mathbf{r}, t) = \left| \Psi_0(\mathbf{r}, t) \right|^2$$

(see Allan Griffin's lectures)

GP

$$i\hbar \frac{\partial}{\partial t} \Psi_0(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left|\Psi_0(\mathbf{r}, t)\right|^2 \right] \Psi_0(\mathbf{r}, t)$$

Now, some remarks on this equation…

GP $$i\hbar \frac{\partial}{\partial t}\Psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi_0(\mathbf{r},t)|^2\right]\Psi_0(\mathbf{r},t)$$

Remark #1

The **GP** equation for order parameter is analog to **Maxwell** equations in classical electrodynamics.

The **condensate** wave function represents the **classical limit of de Broglie waves** (corpuscolar nature of matter no longer important).

Difference : GP contains Planck constant explicitly, Maxwell doesn't.
This follows from the **different dispersion law** of photons and atoms:

from particles to waves: $p \rightarrow \hbar k, E \rightarrow \hbar\omega$

photons $E = cp$ $\longrightarrow$ $\omega = ck$

atoms $E = p^2/2m$ $\omega = \hbar k^2/2m$

particles (energy)   waves (frequency)

GP
$$i\hbar \frac{\partial}{\partial t} \Psi_0(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\left| \Psi_0(\mathbf{r}, t) \right|^2 \right] \Psi_0(\mathbf{r}, t)$$

Remark #2

GP equation is **nonlinear**.

It is a special case of **nonlinear Schrödinger equation (NLSE)** widely used in many fields. It shares many analogies with the physics of **nonlinear optics**.

The equation for the order parameter is **not** an equation for a **wave function** and the solution is **not** a wave function in the usual QM sense (e.g., no linear superposition) . It is sometimes called **macroscopic wave function** or **condensate wave function**.

The order parameter must not be confused with the many-body wave function
$$\Psi_N(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N; t)$$

The nonlinearity comes from interparticle interactions, which are represented by a **mean-field** potential energy in the effective Hamiltonian.

GP $$i\hbar \frac{\partial}{\partial t} \Psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g \left| \Psi_0(\mathbf{r},t) \right|^2 \right] \Psi_0(\mathbf{r},t)$$

Remark #3

The solution is a complex function, with **modulus** and **phase**.

The energy of the system does not depend on phases, but the condensate has a well defined phase. This is can be viewed as an example of **broken gauge symmetry**.

Be careful: in **finite systems** neither Long Range Order or Broken Gauge Symmetry are applicable concepts, strictly speaking, but the order parameter of the condensate is still well defined (eigenfunction of the one-body density matrix).

The **phase** of the order parameter is crucial for **superfluidity**.

GP

$$i\hbar \frac{\partial}{\partial t} \Psi_0(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi_0(\mathbf{r}, t)|^2 \right] \Psi_0(\mathbf{r}, t)$$

Remark #4

The GP equation can be also obtained from the least action principle

$$\delta \left[-i\hbar \int \Psi_0^* \frac{\partial}{\partial t} \Psi_0 \, d\mathbf{r}dt + \int E \, dt \right] = 0$$

which yields

$$i\hbar \frac{\partial}{\partial t} \Psi_0 = \frac{\delta E}{\delta \Psi_0^*}$$

where

$$E = \int d\mathbf{r} \left(\frac{\hbar^2}{2m} |\nabla \Psi_0(\mathbf{r})|^2 + V_{ext}(\mathbf{r})|\Psi_0(\mathbf{r})|^2 + \frac{g}{2} |\Psi_0(\mathbf{r})|^4 \right)$$

Gross-Pitaevskii energy functional

What next:

Statics and dynamics of BECs with GP equation