Bose-Einstein Condensation : production, fundaments and modern aspects

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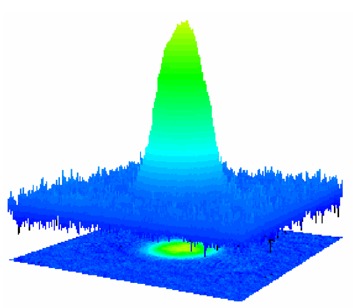
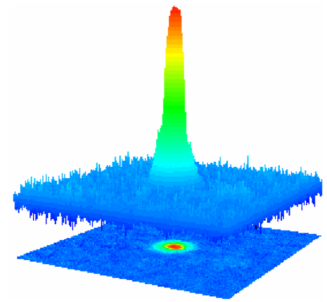
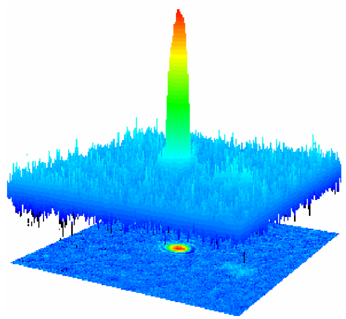


$T > T_c$



$T < T_c$



$T \ll T_c$

Lecture II

Lectures:

1) The importance of BEC. Basic concepts for BEC

2) Interactions in BEC/lattices (exercise) /Thermodynamics

3) How to make BEC?

4) Coherent modes : equivalence with quantum optics

5) Superfluidity aspects: vortices, turbulence and more

V - Interactions

First approx. $H = \frac{p^2}{2m} + U(\bar{r}) + \gamma\, n(\bar{r})$

$\gamma > 0 \longrightarrow$ repulsion

$\gamma < 0 \longrightarrow$ attraction

$$\rho(\epsilon)\,d\epsilon = c^{te} \int_{V^*(r)} \sqrt{\epsilon - U_{eff}(r)}\; d^3r\, d\epsilon$$

Using $n(\bar{r}) \cong \dfrac{e^{-U(r)/kT}}{\lambda_D^3(T)}$

$$\Downarrow$$

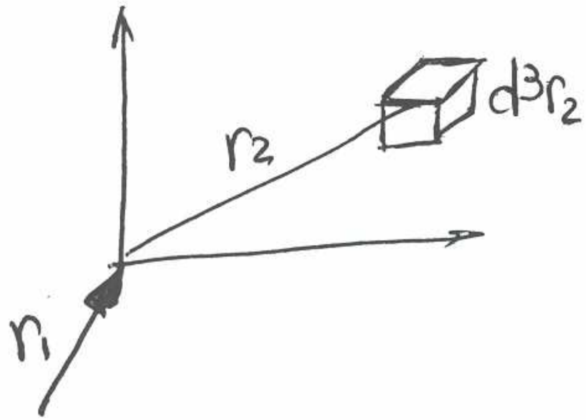
Determine new T_c

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^0}\right)^{\eta+1} - \gamma A \left(\frac{T}{T_c^0}\right)^{\eta+3/2}$$

What is the value of μ?

Consider we know $\underline{\underline{a}}$ and a contact potential

$$V(\bar{r}_1 - \bar{r}_2) = V_0\, \delta(\bar{r}_1 - \bar{r}_2)$$



$$V_0 = \text{Energia} \cdot \text{Volume}$$

$$\text{Energy/particle} = \frac{1}{2} \int V(r_1 - r_2)\, n\, d^3r_2$$

$$= \frac{1}{2} \int V_0\, \delta(r_1 - r_2)\, n\, d^3r_2$$

$$= \frac{1}{2} V_0 n$$

$$\gamma n = \frac{1}{2} V_0 n$$

$$\gamma = \frac{1}{2} V_0$$

From simple scattering using a contact potential $V_0 = \frac{4\pi\hbar^2}{m} a$

$$\longrightarrow \boxed{\gamma = \frac{2\pi\hbar^2}{m} a}$$

Ex: Harmonic Oscillator

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^0}\right)^2 - 4\frac{a}{\lambda_D(T)}\left(\frac{T}{T_c^0}\right)^{7/2} + \cdots$$

$$\Downarrow$$

Calculation of properties.

Gross - Pitaevskii equation

Cooling down the system so that all atoms are in a coherent condensate state

$$\Psi = \prod_{1}^{N} \phi_i$$

$$H = \frac{-\hbar^2}{2m} \nabla^2 + U(\bar{r}) + A|\varphi|^2$$

is the one particle hamiltonian

In a variational approach.

$$E[\varphi] = \langle \varphi | H | \varphi \rangle$$

$$= \int dr \left[\frac{\hbar^2}{2m} |\nabla \varphi|^2 + U(r) |\varphi|^2 + \frac{1}{2} A |\varphi|^4 \right]$$

Minimizing $E[\varphi]$ with the constraint

$\delta E - \mu \delta N = 0$

$\rightarrow$ Lagrange multipliers

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \varphi + U(r)\, \varphi(r) + A |\varphi|^2 = \mu \varphi}$$

Time independent Gross-Pitaevskii

* Has the form of a Schrödinger equation

* Eigenvalue is the chemical potential, not the energy/particle as the conventiona Schrödinger equation $\left(\mu = \frac{\partial E}{\partial N} \right)$

$$E = \langle \varphi | H | \varphi \rangle$$

$\Rightarrow$ Many ways to solve G.P. equation

$(N-1)$

$$A \cong \frac{4\pi N \hbar^2 a}{m}$$

$\boxed{\text{Thomas - Fermi}} \rightarrow$

* Neglecting the kinetic energy term
* Strong repulsive interaction

$$\left[U(r) + A|\varphi|^2 \right] \varphi = \mu \varphi$$

$$\rightarrow \quad |\varphi|^2 = n(r) = \frac{\mu - U(r)}{A}$$

For $U(r) = \mu \longrightarrow$ boudary of the cloud

$\varphi = 0$ outside.

For a <u>Harmonic Trap</u>: $U(r) = \frac{1}{2} m\omega^2 r^2$

and $\int |\varphi|^2 d^3r = 1$

$\longrightarrow \frac{1}{2} m \omega^2 R^2 = \mu \longrightarrow R^2 = \frac{2\mu}{m\omega^2}$

Using $\int_0^R |\varphi|^2 d^3r = 1 \rightarrow \int_0^{\left(\frac{2\mu}{m\omega^2}\right)^{1/2}} \left(\mu - \frac{1}{2} m\omega^2 r\right) \frac{4\pi r^2 dr}{A} = 1$

$\longrightarrow$ extract $\mu = \frac{(15)^{2/5}}{2} \left(\frac{Na}{l_0}\right)^{2/5} \hbar\omega$

$l_0 = \sqrt{\frac{\hbar}{m\omega}}$

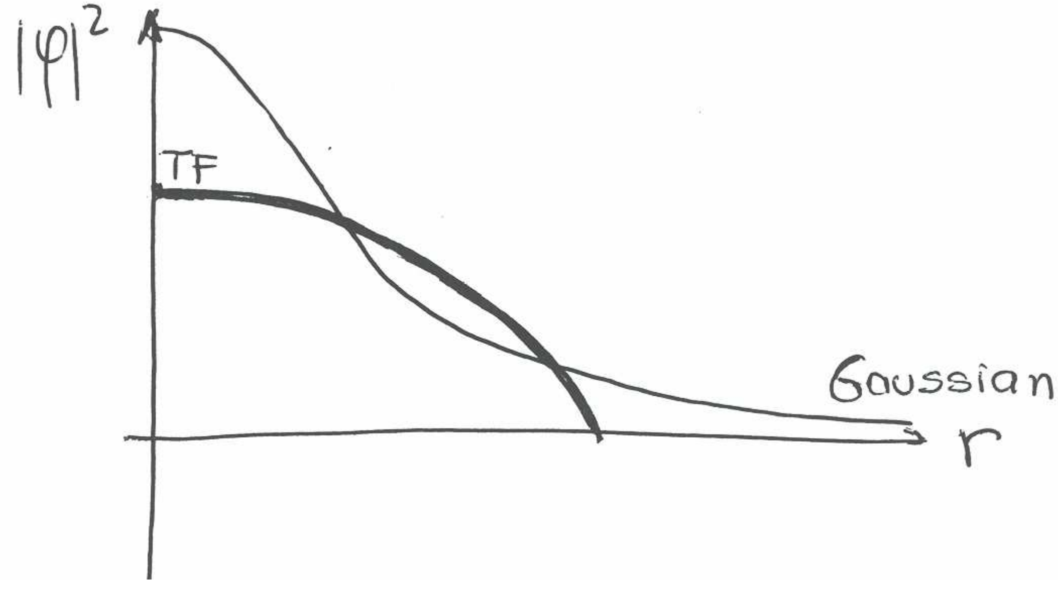
Since $\mu = \dfrac{\partial E}{\partial N} \longrightarrow E \cong \dfrac{5}{7} C' N^{7/5}$

same cte appears in μ

$$\frac{E}{N} \cong \frac{5}{7} C' N^{2/5}$$

$$\Longrightarrow \boxed{\frac{E}{N} = \frac{5}{7}\mu}$$

Energy of each particle in the system



Central density of the cloud at TF is

$$n(0) = \frac{\mu}{A} = \frac{\left(\frac{15}{2}\right)^{2/5}\left(\frac{Na}{l_0}\right)^{2/5}\hbar\omega}{\frac{4\pi \cancel{N} a \hbar^2}{m}}$$

$$n(0) = C\,\frac{\hbar\omega}{(Na)^{3/5}}$$

Renormalized perturbation theory

* Combination of variational and perturbation techniques

$$U(r) = \frac{1}{2} m \omega^2 r^2$$

$$l_0 = \sqrt{\frac{\hbar}{m\omega}} \quad ; \quad a \equiv \text{scatt. length}$$

$$H = -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m \omega^2 r^2 + A|\varphi|^2$$

Defining the interacting parameter g

$$g = \frac{mA}{\hbar^2 l_0} = 4\pi N \frac{a}{l_0}$$

Using length in units of l_0
Energy " " of $\hbar\omega$

$$\boxed{H = -\frac{1}{2}\nabla^2 + \lambda^2 r^2 + g|\varphi|^2}$$

$\lambda \to$ normalized frequency.

Question: How different is this problem from
$$H_0 = -\frac{1}{2}\nabla^2 + b^2 r^2 \quad ?$$

$\longrightarrow$ Solve in terms of H_0

Solution $H_0 \Longrightarrow \{\psi_n\}$ and $\{E_n^0\}$

$$\Delta H = H - H_0 = \frac{1}{2}(\lambda^2 - b^2)\, r^2 + g|\varphi|^2$$

Consider φ as ψ_n

$$\Delta E_n = \langle \psi_n | \Delta H | \psi_n \rangle$$

$$\downarrow$$

$$E_n = E_n^0 + \Delta E_n$$

$$\downarrow$$

$\dfrac{\partial E_n}{\partial b} = 0 \Rightarrow$ equation for b

Determine best b

$$\downarrow$$

Determine best ψ, E_n

$$\Downarrow$$

Minimum $E_n \longrightarrow$ <u>solution</u>

Ground state for harmonic oscillator

$$g_0 = \frac{g}{(2\pi)^{3/2}}$$

$$\Downarrow$$

$$E_0 \cong \left[\frac{3}{2} + g_0 - \frac{3}{4} g_0^2\right] \hbar\omega$$

weakly - coupling

Interesting situation $\longrightarrow$ $\boxed{a < 0}$

Instabilities

Solving for E_o we find that for a critical value of g_o, E_o become complex !!!!

$$\boxed{g_o^c \cong -0.2674}$$

$g_o > g_o^c$
$\Rightarrow$ complex

$$4\pi N \frac{a}{l_o} < g_o^c \Longrightarrow$$

$$N_c \lesssim \frac{(g_o^c)\, l_o}{4\pi a}$$

$N > N_c \longrightarrow$ No stationary solution

Ex: ^7Li $\longrightarrow$ $a = -1.5\ 10^{-7}$ a.u.

$l_0 = 3.16\ 10^{-4}$ a.u.

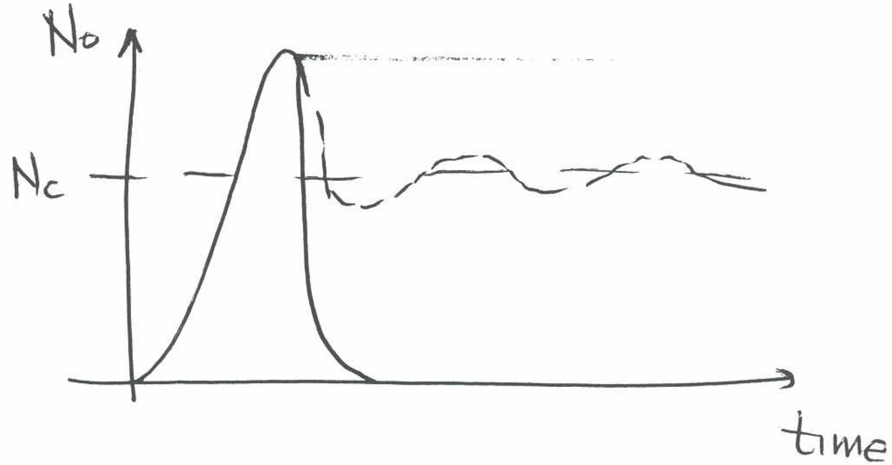
$\longrightarrow$ $\boxed{N_c \cong 707}$

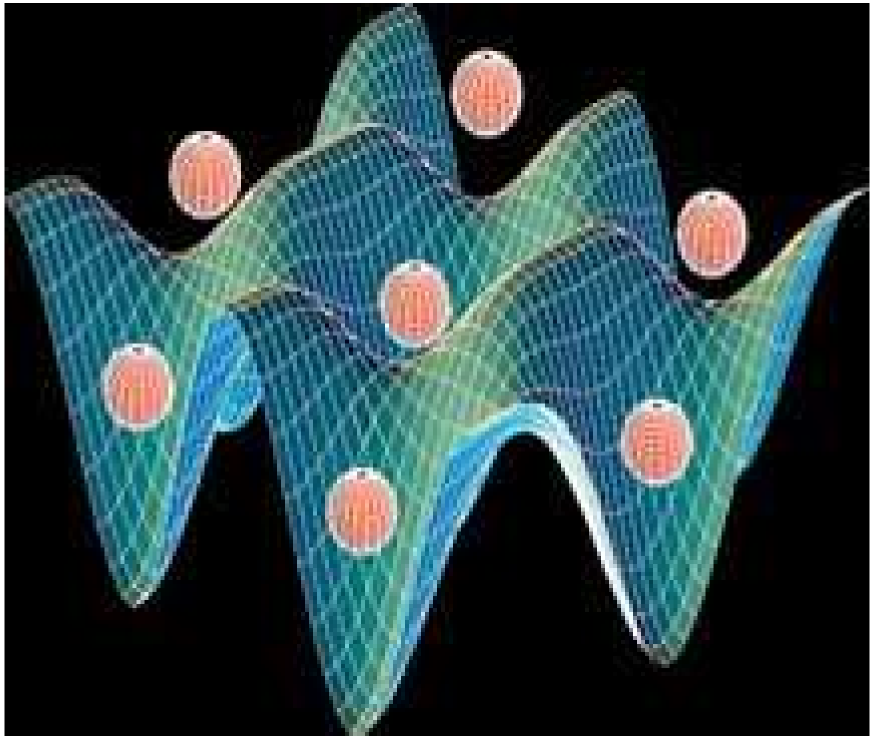
$$E_0 = Re\{E_0\} + Im\{E_0\}\, i$$

$\Uparrow$ decay time

$$\tau \cong \frac{0.9601}{\omega\, g^{2/5}}$$

Hule tis case $\tau \sim 0.6$ ms





BEC from an optical lattice (2D)

Let's consider a set of bosons in a 2D optical Lattice

* Consider atoms cooled to the lowest vibrational states of the lattice sites.
* There is no more than one atom per site
* Occupation number is ξ.
* ~~Coolin~~ Adiabatically remove the lattice potentials
* If $\xi = 1 \Longrightarrow$ atoms became a $T=0$ BEC
* If $\xi < 1$. What is the smallest occupation number ξ_0 which will result in formation of BEC after adiabatically removing the lattice

During adiabatically change of the lattice, the entropy $dS = \frac{dQ}{T}$ dos not change. Therefore, to end with a BEC after relaxing the optical lattice one must have in the lattice an entropy equivalent to the free BEC entropy.

For the lattice: Consider that there are P available lattice sites, and ξP occupied The entropy of this system can be calculated evaluating the configurations of occupation

$$\Omega = \frac{P!}{(\xi P)!\,(P-\xi P)!}$$

The entropy is calculated as

$$S = K_B \ln \Omega$$

$$= K_B \ln \frac{P!}{(\xi P)!\,(P-\xi P)!}$$

$$= K_B \left\{ \ln P! - \ln (\xi P)! - \ln (P-\xi P)! \right\}$$

Using Stirling's relation $\left(\ln P! \approx P \ln P - P \right)$

$$\boxed{S = K_B P \left\{ (\xi - 1) \cdot \ln (1-\xi) - \xi \ln \xi \right\}}$$

For Bose gas below T_c

$$C_v = \left(\frac{\partial E}{\partial T}\right)_V \quad \text{and} \quad S = \int_0^T \frac{C_v\, dT'}{T'}$$

To evaluate E: $\quad E = -\frac{\partial}{\partial \beta} \ln Z$

$$\Downarrow$$

$$E = \frac{3}{2} \frac{KT}{\lambda_D^3(T)} g_{5/2}\left(e^{\mu/kT}\right)$$

$$\Downarrow$$

$$S = 1.283\, N \left(\frac{T}{T_c}\right)^{3/2}$$

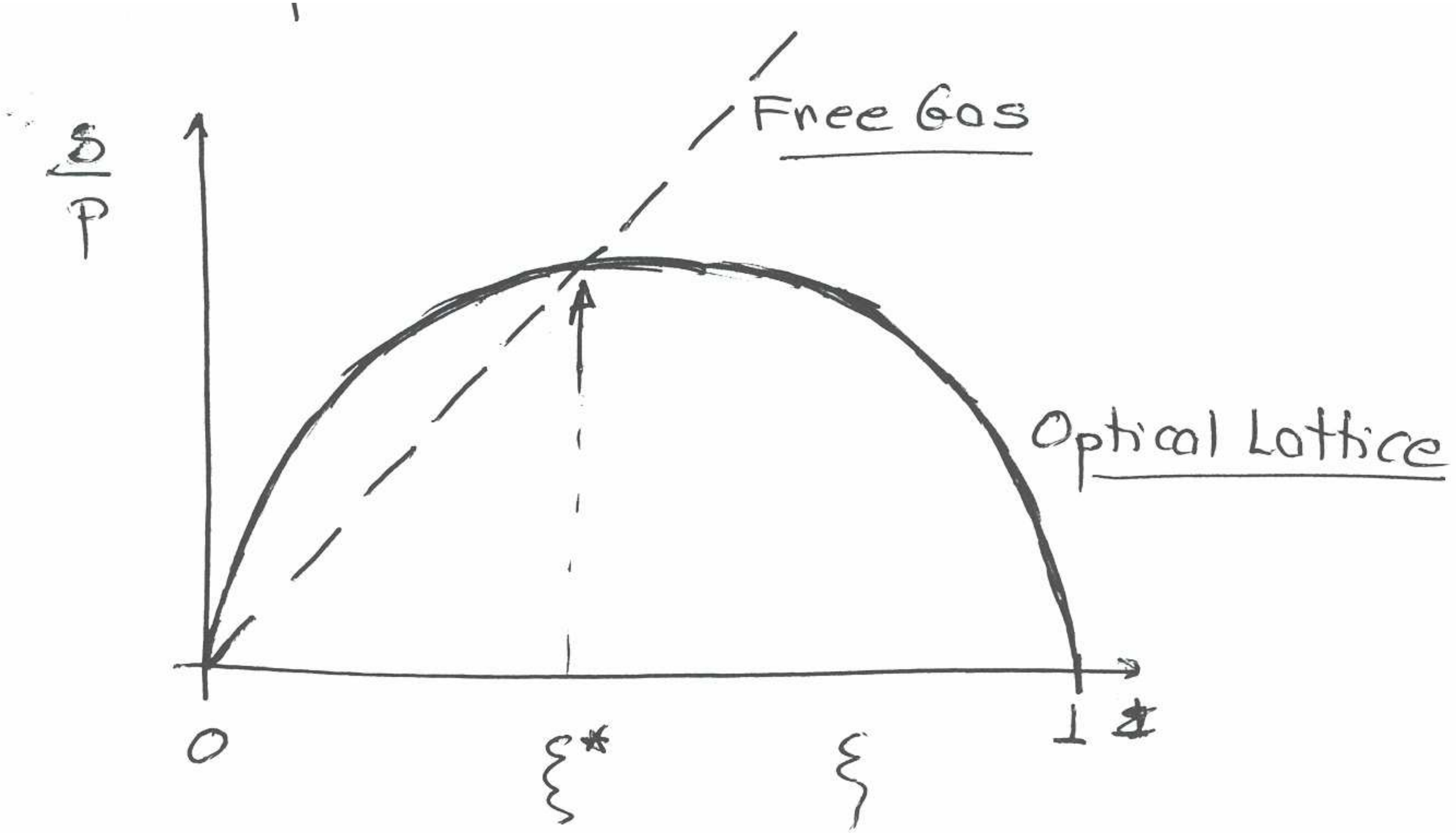
For the lattice case, $N = \xi P$

so $S = 1.283\, \xi P \left(\frac{T}{T_c}\right)^{3/2}$

Per lattice site

$$\frac{S}{P} = K_B \left\{ (\xi - 1) \ln(1-\xi) - \xi \ln \xi \right\} \quad \text{lattice}$$

$$\frac{S}{P} = 1.283\, \xi \left(\frac{T}{T_c}\right)^{3/2}$$



$\xi > \xi^* \approx 0.54$ to keep having a BEC

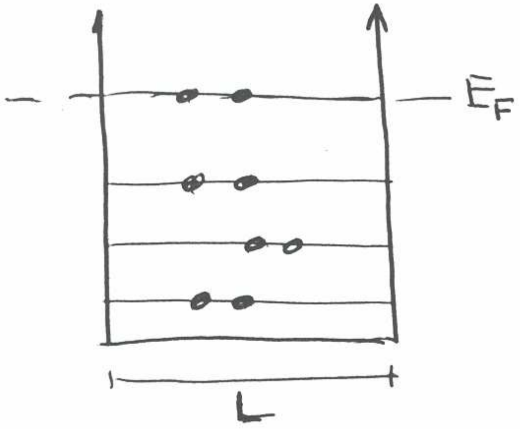
You will cool down with the transformation

Fermi Energy for a Harmonic Trap

* Many groups working with fermions

A collection of fermions in a box (3D)

(Consider Spin ½)



$$E = \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + n_2^2 + n_3^2\right)$$

$$\Downarrow$$

$$n_1^2 + n_2^2 + n_3^2 = R^2 = \frac{2mE_F\,L^2}{\hbar^2\pi^2}$$

$$N = 2 \times \frac{1}{8}\,\frac{4\pi}{3}R^3 \Rightarrow \boxed{E_F = \frac{\hbar^2\pi^2}{2m}\left(\frac{3n}{\pi}\right)^{2/3}}$$

$$n = \frac{N}{V}$$

N particles in a Harmonic Trap.

$$\mathcal{U}(x,y,z) = \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$$

$$\rightarrow \quad E = \hbar\omega (n_x + n_y + n_z) \qquad (E_0 \equiv 0)$$

In units of $\hbar\omega$

$$E = n_x + n_y + n_z$$

For a given energy, how many combinations of n_x, n_y, n_z are possible?

Let's count: <u>For a energy E</u>

M_x	M_y	M_z	# states
E	0	0	1
$E-1$	1	0	2
$E-1$	0	1	
$E-2$	2	0	3
$E-2$	1	1	
$E-2$	0	2	
	$\vdots$		$\vdots$

Number of states for a given energy E

$$M(E) = \sum_{i=1}^{E} (i+1)$$

$$= \left(\frac{E+2}{2}\right)(E+1) \approx \frac{E^2}{2}$$

To find Fermi-Energy

$$N = \sum_{E=1}^{E_F} M(E) \cong \frac{1}{2} \sum_{E=1}^{E_F} E^2 = \frac{1}{2} \frac{E_F(1+E_F)(1+2E_F)}{6}$$

$$N \approx \frac{E_F^3}{6}$$

$$\therefore \quad E_F = \hbar\omega\,(6N)^{1/3}$$

Bosons

Temperature

$$\boxed{KT_F = E_F}$$

$$\left(KT_c = 0.94\,\hbar\omega N^{1/3} \right)$$

If an unisotropy is introduced

$$U(x,y,z) = \frac{1}{2} m\omega^2 \left((\lambda x)^2 + y^2 + z^2 \right)$$

$$\Downarrow$$

Easy to show that

$$E_F = \hbar\omega \left(6N\lambda \right)^{1/3}$$

Important properties of Quant. Fluids

* Superfluidity
* Coherence

(1) Speed of sound : travel of compression waves

v_s depends on : μ, M and ω

Dimensional analysis : $v_s = \mu^\alpha M^\beta \omega^\gamma$

$$v_s \cong \sqrt{\frac{\mu}{M}}$$

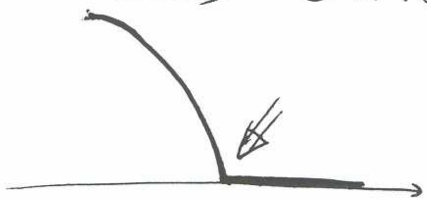
Motions slower than v_s – condensate flows smoothly around obstacles.

– Above v_s may occur excitation of particles out of the condensate.

(2) Healing length

Neglecting kinetic term $\longrightarrow$ Thomas-Fermi

$\longrightarrow$ unrealistic sharp edge



$$\nabla^2 \psi \to \infty$$

$\Rightarrow$ At the boundary, have to take into account kinetic energy

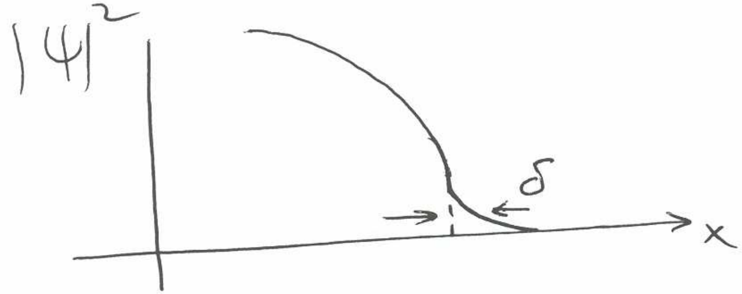
$\Rightarrow$ Shortest distance over which wavefunction can change $\longrightarrow$ healing length (ξ)

Kinetic term = $\dfrac{\hbar^2}{2M\xi^2}$ and this must be about the energy amplitude of the system

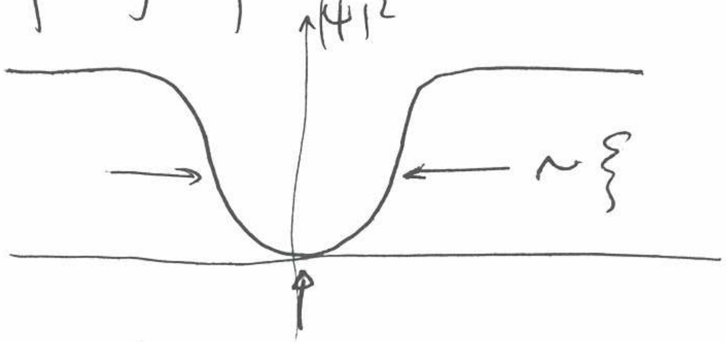
$$\frac{\hbar^2}{2M\xi^2} \simeq \mu = \frac{4\pi\hbar^2 a\, n_0}{M}$$

$$\rightarrow \quad \boxed{\xi = \frac{1}{\sqrt{8\pi a\, n_0}}}$$

Typical for Na BEC $\xi \sim 0{,}3\,\mu m$



The healing length ξ is also the quantity that determines the size of <u>vortices</u> when the confining potential rotates.



" ξ is the typical length in which the superfluid recovers from a sharp change"